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ABSTRACT

We discuss the conditional expectations and martingales in relevance with G-strongly quasi-invariant states on a C^*-algebra $\mathcal{A}$, where G is a separable locally compact group of $*$-automorphisms of $\mathcal{A}$. In the von Neumann algebra $\mathfrak{A}$ of the GNS representation, we define a unitary representation of the group and a group $\widehat{G}$ of $*$-automorphisms of $\mathfrak{A}$, which is homomorphic to G. For the case of compact G, we find a $\widehat{G}$-invariant state on $\mathfrak{A}$ and define a conditional expectation with range the $\widehat{G}$-fixed subalgebra. When G is the union of increasing compact groups, we construct a sequence of conditional expectations and thereby construct (backward) martingales, which have limits by the martingale convergence theorem. As an example we consider S_∞ the group of local permutations which acts on a C^*-algebra of infinite tensor product of finite dimensional C^*-algebras. We also find an application in classical spin systems.

I. INTRODUCTION

Since it was introduced by Umegaki,[16] the conditional expectation in an operator algebra is an important concept not just as an extension of classical concept but also as a tool to characterize the structure of the operator algebra. The concept has been further developed,[1,14] and particularly it is a key concept to consider the martingales.[5,9,15]

Given a von Neumann algebra $\mathfrak{A}$ with a faithful normal state ρ, suppose that $(\mathfrak{B}_n)_n$ is an increasing (or decreasing, respectively) sequence of von Neumann subalgebras of $\mathfrak{A}$ and $(E_n)_n$ is a sequence of conditional expectations E_n onto $\mathfrak{B}_n$ for each n. A sequence $(x_n)_n$ of $\mathfrak{A}$ is called martingale if (i) $x_n \in \mathfrak{B}_n$, (ii) $E_m(x_n) = x_m$ [respectively $E_n(x_m) = x_n$ in the decreasing case] for $m \le n$. The martingale convergence theorem says that if $(x_n)_n$ is a martingale, then there is an $x \in \mathfrak{A}$ such that $x_n \to x$ strongly as $n \to \infty$.[3,5,9–11,15]

In this paper we first investigate the martingale theory in relevance with G-strongly quasi-invariant states on a C^*-algebra $\mathcal{A}$. Here G is a compact group or a separable locally compact group, which is the union of increasing compact groups of $*$-automorphisms of $\mathcal{A}$. The main point lies in finding a state (weight) which is invariant under the group actions.

Let φ be a faithful state on a C^*-algebra $\mathcal{A}$ and G be a group of $*$-automorphisms of $\mathcal{A}$. We say that φ is G-strongly quasi-invariant on $\mathcal{A}$ if for all $g \in G$, there exists a self-adjoint operator $x_g \in \mathcal{A}$ such that[2]

$$\varphi(g(a)) = \varphi(x_g a), \qquad \forall a \in \mathcal{A}.$$

In this case, it is proved in Ref. 2 that for each $g \in G$, x_g is strictly positive and it is an element of the centralizer of φ. Moreover, the algebra generated by the elements x_g is a commutative C^*-algebra.

First, suppose G is compact. In the von Neumann algebra of the representation of $(\mathcal{A}, \varphi)$, $\mathfrak{A} := \pi(\mathcal{A})''$, we define a unitary representation of G and using this we induce a group $\widehat{G}$, which is homomorphic to G, of $*$-automorphisms of $\mathfrak{A}$. Moreover, we find a $\widehat{G}$-invariant state ψ_G on $\mathfrak{A}$. Then we construct a conditional expectation $E : \mathfrak{A} \to \mathfrak{B}$, where $\mathfrak{B}$ is a von Neumann subalgebra of $\mathfrak{A}$ consisting of $\widehat{G}$-fixed elements. Recently, in (Ref. 7, Theorem 3.2, Theorem 3.3), a characterization of G-quasi-invariant states with bounded cocycles was done by a G-invariant state and a bounded invertible operator.

When G is a separable locally compact group, in particular G has the structure $G = \bigcup_{N\in\mathbb{N}} G_N$, where $(G_N)_{N\in\mathbb{N}}$ is an increasing sequence of compact groups of $*$-automorphisms of $\mathcal{A}$, imposing a condition that supports the existence of $\widehat{G}$-invariant state, we construct a sequence of conditional expectations $(E_N)_N$ with ranges $\mathfrak{B}_N$'s, where for each N, $\mathfrak{B}_N$ is a $\widehat{G}_N$-invariant subalgebra of $\mathfrak{A}$. The sequence $(E_N)_N$ satisfies the martingale property, namely, for all $M \le N$,

$$E_N E_M (= E_M E_N) = E_N.$$

By this we have martingales: for any $x \in \mathfrak{A}$, the sequence $(x_n)_n$ with $x_n := E_n(x)$ is a (backward) martingale. Then by a martingale convergence theorem such a sequence converges strongly to an element. This enables us to have a limit $E_N \to E_\infty$ as $N \to \infty$, where E_∞ is an Umegaki conditional expectation onto $\mathrm{Fix}(u_G)$, the $\widehat{G}$-invariant von Neumann subalgebra of $\mathfrak{A}$. Here, the convergence is in the pointwise strong sense, i.e., $(E_N(x))_N$ converges to $E_\infty(x)$ strongly for every $x \in \mathfrak{A}$ (Theorem 4.2). We will apply these theory to two non-trivial examples. One is the group of local permutations and the other is the classical spin systems.

This paper is organized as follows. In Sec. II, we briefly recall the definition of strongly quasi-invariant states with respect to compact groups. Next, given a G-strongly quasi-invariant state φ, we consider the GNS representation and induce a homomorphic group of automorphisms and an invariant state in the von Neumann algebra of the representation. Then, we construct a conditional expectation. In Sec. III, we discuss the inductive limit of compact groups. A sequence of conditional expectations will be considered. In Sec. IV, we discuss the martingales. Section V is devoted to an example. We consider the locally compact group consisting of the local permutations on the set of nonnegative integers. In the final Sec. VI, we apply the theory to the classical spin systems. We show that under certain conditions the Gibbs measures are G-strongly quasi-invariant, where G is a locally compact group of spin flips or spin exchanges.

II. STRONGLY QUASI-INVARIANT STATES WITH RESPECT TO A COMPACT GROUP

In this section we consider the G-strongly quasi-invariant states for a compact group of $*$-automorphisms of a C^*-algebra and consider the conditional expectation.

A. G-strongly quasi-invariant states

Here we assume that G is a compact group of $*$-automorphisms of a C^*-algebra $\mathcal{A}$, and φ is a G-strongly quasi-invariant faithful state on $\mathcal{A}$. We assume that the map $G \ni g \mapsto x_g \in \mathcal{A}$ is continuous. Denote by $\{\mathcal{H}, \pi, \Phi\}$ the cyclic representation of $(\mathcal{A}, \varphi)$. By (Ref. 4, Proposition 2.3.1), the map $g \mapsto \pi(x_g)$ and consequently the map $g \mapsto \pi(x_g^{1/2})$ is continuous. From,[2] it is proved that the map U defined by

$$U_g \pi(a)\Phi = \pi\Big(g(a) x_{g^{-1}}^{1/2}\Big)\Phi \ ; \quad \forall\ a \in \mathcal{A}, \tag{2.1}$$

is a unitary representation of G on $\mathcal{H}$. We assume the map $g \mapsto U_g$ is strongly continuous on $\mathcal{H}$, define

$$P_G := \int_G U_g dg,$$

where, and in the sequel, dg denotes the normalized Haar measure on the compact group under consideration.

Lemma 2.1 P_G is an orthogonal projection on $\mathcal{H}$ with range

$$P_G(\mathcal{H}) = \{\xi \in \mathcal{H} : \quad U_g(\xi) = \xi, \quad \forall g \in G\} =: \mathrm{Fix}_G(\mathcal{H}). \tag{2.2}$$

Proof. We have

$$\begin{aligned} P_G^2 &= \int_G \left(\int_G U_g U_h dh \right) dg \\ &= \int_G \left(\int_G U_{gh} dh \right) dg \\ &= \int_G \left(\int_G U_h dh \right) dg \qquad \text{(The Haar measure is left translation invariant)} \\ &= \int_G P_G dg = P_G. \end{aligned}$$

On the other hand, since the Haar measure is invariant by inversion, one gets

$$P_G^* = \int_G U_g^* dg = \int_G U_{g^{-1}} dg = \int_G U_g dg = P_G.$$

Now for any $\xi \in \mathcal{H}$ and $g \in G$,

$$U_g P_G(\xi) = \left(\int_G U_g U_h dh\right)(\xi) = \left(\int_G U_{gh} dh\right)(\xi) = \left(\int_G U_h dh\right)(\xi) = P_G(\xi).$$

So, $P_G(\mathcal{H}) \subset \mathrm{Fix}_G(\mathcal{H})$. Conversely, suppose $\xi \in \mathrm{Fix}_G(\mathcal{H})$. Then,

$$P_G(\xi) = \int_G U_g(\xi) dg = \int_G \xi dg = \xi.$$

Hence $\mathrm{Fix}_G(\mathcal{H}) \subset P_G(\mathcal{H})$. We conclude that P_G is an orthogonal projection onto $\mathrm{Fix}(G)$. □

From (2.1) it follows that

$$U_g \Phi = \pi\left(x_{g^{-1}}^{1/2}\right)\Phi.$$

It holds that

$$\Phi_G := P_G \Phi = \left(\int_G U_g dg\right)\Phi = \int_G \pi\left(x_{g^{-1}}^{1/2}\right) dg \Phi = \int_G \pi\left(x_g^{1/2}\right) dg \Phi. \tag{2.3}$$

Let us define the operator appearing on the r.h.s. of (2.3) by

$$K_G := \int_G \pi\left(x_g^{1/2}\right) dg. \tag{2.4}$$

The operator K_G will play a central role in this paper, and we emphasize here that the operators P_G and K_G, both acting on $\mathcal{H}$, are not equal to each other in general but result in the same vector Φ_G when applied to the vector Φ. See Example 2.4 below. Moreover, by (2.2) Φ_G is $U(G)$-invariant where $U(G) = \{U_g : g \in G\}$:

$$U_g \Phi_G = \Phi_G, \quad g \in G. \tag{2.5}$$

From the continuity of the map $g \mapsto \pi(x_g^{1/2})$ one can show that the operator K_G is bounded with a bounded inverse (see e.g., the proof of [Ref. 2, Theorem 1]). Therefore,

$$\Phi_G = P_G \Phi = K_G \Phi \neq 0.$$

And for any $a \in \mathcal{A}$, one has

$$\varphi(a) = \langle \Phi,\ \pi(a)\Phi \rangle = \langle K_G^{-1}\Phi_G,\ \pi(a) K_G^{-1}\Phi_G \rangle = \langle \Phi_G,\ K_G^{-1}\pi(a) K_G^{-1}\Phi_G \rangle. \tag{2.6}$$

Now define a state on the von Neumann algebra $\mathfrak{A} := \pi(\mathcal{A})''$ by

$$\psi_G(x) := \frac{1}{\|\Phi_G\|^2}\langle \Phi_G, x\Phi_G \rangle, \quad x \in \mathfrak{A}. \tag{2.7}$$

For each $g \in G$, define a linear $*$-map on $\mathfrak{A}$ by

$$u_g(x) := U_g x U_g^*, \quad x \in \mathfrak{A}. \tag{2.8}$$

In particular u_g acts on $\pi(\mathcal{A})$ as

$$u_g(\pi(a)) = \pi(g(a)), \quad a \in \mathcal{A}. \tag{2.9}$$

In fact, by using the cocycle property of x_g's we have $x_g^{-1} = g^{-1}(x_{g^{-1}})$.[2] Thus, for all $a, b \in \mathcal{A}$,

$$\begin{aligned} u_g(\pi(a))\pi(b)\Phi &= U_g \pi(a) U_g^* \pi(b)\Phi \\ &= U_g \pi(a)\pi(g^{-1}(b) x_g^{1/2})\Phi \\ &= \pi\left(g(a) b g(x_g^{1/2}) x_{g^{-1}}^{1/2}\right)\Phi \\ &= \pi(g(a))\pi(b)\Phi. \end{aligned}$$

Therefore, by letting $\widehat{G} := \{u_g : g \in G\}$, we see that $\widehat{G}$ is a group of $*$-automorphisms of $\mathfrak{A}$, which is homomorphic to G.

Proposition 2.2 The state ψ_G on $\mathfrak{A}$ defined in (2.7) is $\widehat{G}$-invariant.

Proof. Take arbitrary $g \in G$ and $a \in \mathcal{A}$. Putting $\Psi_G(\cdot) := \|\Phi_G\|^2 \psi_G(\cdot)$, by (2.5),

$$\begin{aligned}\Psi_G(u_g(\pi(a))) &= \langle \Phi_G, U_g \pi(a) U_g^* \Phi_G\rangle \\ &= \langle U_{g^{-1}}\Phi_G, \pi(a) U_{g^{-1}}\Phi_G\rangle \\ &= \langle \Phi_G, \pi(a)\Phi_G\rangle = \Psi_G(\pi(a)).\end{aligned}$$

Since $\pi(\mathcal{A})$ is weakly dense in $\mathfrak{A}$, the proof is completed. □

Remark 2.3 If we define a state φ_G on $\mathcal{A}$ by

$$\varphi_G(a) := \psi_G(\pi(a)), \quad a \in \mathcal{A}, \tag{2.10}$$

then by Proposition 2.2, φ_G is a G-invariant state. In fact,

$$\begin{aligned}\varphi_G(g(a)) &= \psi_G(\pi(g(a))) \\ &= \psi_G(u_g(\pi(a))) \\ &= \psi_G(\pi(a)) \quad \text{(by Proposition 2.2)} \\ &= \varphi_G(a).\end{aligned}$$

Example 2.4 Let us consider the Example 2 in Ref. 6. Let $\mathcal{A} = \mathcal{M}_2(\mathbb{C})$. Let $G = \{g_\theta | \theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\}$, where

$$g_\theta(a) := R_{-\theta} a R_\theta, \quad a \in \mathcal{A}, \quad R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ sin\theta & \cos\theta \end{pmatrix}. \tag{2.11}$$

Let

$$\rho = \begin{pmatrix} \frac{e^\beta}{1+e^\beta} & 0 \\ 0 & \frac{1}{1+e^\beta} \end{pmatrix}.$$

The state $\varphi(a) := \mathrm{tr}(\rho a)$, $a \in \mathcal{A}$, is G-strongly quasi-invariant with

$$x_{g_0} = x_{g_\pi} = I, \quad x_{g_{\pi/2}} = x_{g_{3\pi/2}} = \begin{pmatrix} e^{-\beta} & 0 \\ 0 & e^\beta \end{pmatrix}. \tag{2.12}$$

It is not hard to see that in the GNS representation $(\mathcal{H}, \pi, \Phi)$ of $(\mathcal{A}, \varphi)$ [$\mathcal{H} = \mathcal{M}_2(\mathbb{C})$, $\Phi = I_2$, the 2×2 unit matrix, and we use $\pi(a) = a$ for simplicity], the unitary operators U_g are computed concretely by the definition $U_g(\pi(a)\Phi) = \pi(g(a)x_{g^{-1}}^{1/2})\Phi$ and the formula (2.12), and then the projection P_G acts as

$$P_G\pi(a)\Phi = \frac{1}{2}\begin{pmatrix} a_{11} + e^{-\beta/2}a_{22} & a_{12} - e^{\beta/2}a_{21} \\ a_{21} - e^{-\beta/2}a_{12} & a_{22} + e^{\beta/2}a_{11} \end{pmatrix}, \quad a = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}. \tag{2.13}$$

Obviously, there is no $P \in \mathcal{A}$ such that $P_G\pi(a)\Phi = \pi(Pa)\Phi$ for all $a \in \mathcal{A}$, which means that $P_G \notin \pi(\mathcal{A})$. On the other hand, one can directly show that

$$K_G = \pi\left(\int_G x_g^{1/2} dg\right) = \frac{1}{2}\begin{pmatrix} 1 + e^{-\beta/2} & 0 \\ 0 & 1 + e^{\beta/2} \end{pmatrix} \in \pi(\mathcal{A}). \tag{2.14}$$

One sees, however, that the two operators P_G and K_G act on $\Phi = I_2$ resulting in

$$P_G\Phi = K_G\Phi = \frac{1}{2}\begin{pmatrix} 1 + e^{-\beta/2} & 0 \\ 0 & 1 + e^{\beta/2} \end{pmatrix} \equiv \Phi_G \in \mathcal{H}.$$

B. Conditional expectation

Let G be a compact group of $*$-automorphisms of a C^*-algebra $\mathcal{A}$ and φ a G-strongly quasi-invariant state on $\mathcal{A}$. We use the same notations introduced in Subsection II A. In particular, the von Neumann algebra $\mathfrak{A} = \pi(\mathcal{A})''$ obtained by a GNS representation of $(\mathcal{A}, \varphi)$ is equipped with a $\widehat{G}$-invariant state ψ_G. Define

$$\mathrm{Fix}(u_G) := \{x \in \mathfrak{A} : \; u_g(x) = x, \quad \forall \, g \in G\}. \tag{2.15}$$

Denoting $\mathfrak{B} := \mathrm{Fix}(u_G)$ a von Neumann subalgebra of $\mathfrak{A}$ we define a $*$-map $E : (\mathfrak{A}, \psi_G) \to \mathfrak{B}$ by

$$E(x) := \int_G u_g(x) dg, \quad x \in \mathfrak{A}. \tag{2.16}$$

It is promptly shown that $E(x) \in \mathfrak{B}$. In fact, for any $g \in G$,

$$u_g(E(x)) = \int_G u_g u_h(x) dh = \int_G u_{gh}(x) dh = \int_G u_h(x) dh = E(x).$$

To show that the map E is an onto map, suppose that $x \in \mathfrak{B}$. Then, $u_g(x) = x$ for all $g \in G$ and hence $E(x) = \int_G u_g(x) dg = \int_G x dg = x$ showing that x is in the range of E.

Theorem 2.5 *The map $E : (\mathfrak{A}, \psi_G) \to \mathfrak{B}$ is an Umegaki conditional expectation.*[16]

Proof. We have to show that E is a normal contractive positive projection satisfying (i) $E(yxy') = yE(x)y'$ for all $x \in \mathfrak{A}$, $y, y' \in \mathfrak{B}$, (ii) $\psi_G \circ E = \psi_G$. Since the map $g \mapsto u_g$ is a $*$-automorphism and G is compact, the normality, contractivity, and positivity are obvious. We check the remaining two properties. Let $x \in \mathfrak{A}$ and $y, y' \in \mathfrak{B}$. Then,

$$E(yxy') = \int_G u_g(yxy') dg = \int_G u_g(y) u_g(x) u_g(y') dg = \int_G y u_g(x) y' dg = yE(x)y'.$$

Also, for any $x \in \mathfrak{A}$, by the $\widehat{G}$-invariance of ψ_G (Proposition 2.2),

$$\psi_G(E(x)) = \psi_G\left(\int_G u_g(x) dg\right) = \int_G \psi_G(u_g(x)) dg = \int_G \psi_G(x) dg = \psi_G(x).$$

The proof is completed. □

III. INDUCTIVE LIMIT OF COMPACT GROUPS

Let $(G_N)_{N \in \mathbb{N}}$ be an increasing sequence of compact groups of $*$-automorphisms of $\mathcal{A}$ and let

$$G := \bigcup_{N \in \mathbb{N}} G_N, \tag{3.1}$$

which is a locally compact group. Let φ be a G-strongly quasi invariant state. Then, obviously φ is G_N-strongly quasi invariant for all N. Let $P_N \equiv P_{G_N}$ be the orthogonal projection onto $\mathrm{Fix}_{G_N}(\mathcal{H})$ as was defined in Sec. II. Then for any $M \le N$, $\mathrm{Fix}_{G_N}(\mathcal{H}) \subset \mathrm{Fix}_{G_M}(\mathcal{H})$ and therefore $P_N \le P_M$, in particular

$$P_N P_M = P_N. \tag{3.2}$$

Since $(P_N)_N$ is a decreasing sequence of orthogonal projections, it converges strongly to an orthogonal projection denoted by P_G:

$$P_G := s - \lim_N P_N.$$

Denote by $\mathcal{H}_\infty$ the range of P_G and recall that

$$\mathrm{Fix}_G(\mathcal{H}) := \{\xi \in \mathcal{H} : \quad U_g(\xi) = \xi, \quad \forall g \in G\}.$$

Lemma 3.1 $\mathcal{H}_\infty = \mathrm{Fix}_G(\mathcal{H})$.

Proof. Notice by Lemma 2.1

$$\mathcal{H}_\infty = \cap_N P_N(\mathcal{H}) = \cap_N \mathrm{Fix}_{G_N}(\mathcal{H}).$$

Thus the proof is completed by just noticing $\mathrm{Fix}_G(\mathcal{H}) = \cap_N \mathrm{Fix}_{G_N}(\mathcal{H})$. □

Denote by $\Phi_N := P_N\Phi$, $\Phi_G := P_G\Phi = \lim_N P_N\Phi = \lim_N K_N\Phi$, where $K_N := K_{G_N}$ is given by (2.4) for $G = G_N$. Throughout the paper we assume the Hypothesis (H) below.

(H) **Hypothesis.** $\Phi_G \neq 0$.

Below in Proposition 3.4 and Proposition 3.5, we give sufficient conditions for (H). Under the hypothesis (H), we define a state ψ_G on $\mathfrak{A}$ as in (2.7): $\psi_G(\cdot) = \frac{1}{\|\Phi_G\|^2}\langle\Phi_G, \cdot\Phi_G\rangle$.

Theorem 3.2 *Suppose that the hypothesis (H) holds. Then the state ψ_G is $\widehat{G}$-invariant.*

Proof. Take any $a \in \mathcal{A}$ and $g \in G$. There exists an $N_g \in \mathbb{N}$ such that $g \in G_N$ for all $N \geq N_g$. Then, for all $N \geq N_g$, by Proposition 2.2 we have

$$\langle\Phi_N, u_g(\pi(a))\Phi_N\rangle = \langle\Phi_N, \pi(a)\Phi_N\rangle.$$

Taking the limit $N \to \infty$ in both sides, we get

$$\langle\Phi_G, u_g(\pi(a))\Phi_G\rangle = \langle\Phi_G, \pi(a)\Phi_G\rangle,$$

that is, $\psi_G(u_g[\pi(a)]) = \psi_G[\pi(a)]$, the $\widehat{G}$-invariance of ψ_G. □

An immediate consequence of Theorem 3.2 and Remark 2.3 is the following.

Corollary 3.3 Under the Hypothesis (H), define a state φ_G on $\mathcal{A}$ by

$$\varphi_G(a) := \psi_G(\pi(a)), \quad a \in \mathcal{A}.$$

Then, φ_G is G-invariant.

Let us now consider some sufficient conditions for the Hypothesis (H). Let λ_N denote the normalized Haar measure on the group G_N for each N.

Proposition 3.4 Suppose that there are constants $\epsilon_0 > 0$, $\delta_0 > 0$, and $N_0 \in \mathbb{N}$ such that for each $N \geq N_0$ there is a subset $A_N \subset G_N$ such that $\lambda_N(A_N) \geq \delta_0$ and for $g, h \in A_N$, $\varphi\Big((x_g x_h)^{1/2}\Big) \geq \epsilon_0$. Then, the Hypothesis (H) holds.

Proof. Recall that x_g's commute with themselves. Let $N \geq N_0$. By the relation $\Phi_N = K_N\Phi$, we have

$$\begin{aligned}\|\Phi_N\|^2 &= \langle K_N\Phi, K_N\Phi\rangle \\ &= \langle\Phi, \iint_{(G_N)^2}\pi\Big((x_g x_h)^{1/2}\Big)dhdg\Phi\rangle \\ &= \iint_{(G_N)^2}\varphi\Big((x_g x_h)^{1/2}\Big)dhdg \\ &\geq \iint_{(A_N)^2}\varphi\Big((x_g x_h)^{1/2}\Big)dhdg \geq \epsilon_0(\delta_0)^2.\end{aligned} \tag{3.3}$$

Here we have used the positivity of $x_g x_h$ since it is a product of commuting positive operators. Since $\Phi_N \to \Phi_G$ strongly, we conclude $\|\Phi_G\| \geq \sqrt{\epsilon_0}\delta_0 > 0$. □

Here we consider another sufficient condition for the Hypothesis (H).

Proposition 3.5 Suppose that the sequence of operators $(K_N)_N$ converges weakly in $\mathcal{B}(\mathcal{H})$ to an invertible operator K_G. Then, Φ_G is nonzero and ψ_G is faithful.

Proof. First we show that $\Phi_G = K_G\Phi$. In fact, for any $\Psi \in \mathcal{H}$,

$$\begin{aligned}\langle\Phi_G - K_G\Phi, \Psi\rangle &= \lim_{N\to\infty}(\langle P_N\Phi, \Psi\rangle - \langle K_N\Phi, \Psi\rangle) \\ &= 0,\end{aligned}$$

since $P_N\Phi = K_N\Phi$ for all N. It proves the claim. Now given that $\Phi_G = K_G\Phi$ and K_G is invertible, it follows that $\Phi_G \neq 0$. Particularly, it also implies that Φ_G is a cyclic vector for $\mathfrak{A}$. By (Ref. 6, Proposition 4.1), $\Phi_g := \pi(\sqrt{x_g})\Phi$ belongs to the positive cone $\mathcal{P}$ associated with Φ. Obviously, $\Phi_N = \int_{G_N}\Phi_g dg$ also belongs to $\mathcal{P}$ and therefore Φ_G is an element of $\mathcal{P}$. Now, by (Ref. 4, Proposition 2.5.30), Φ_G is also separating for $\mathfrak{A}$. Therefore, ψ_G is faithful. □

IV. MARTINGALES

We continue with a setting of the previous section. $G = \cup_N G_N$ is a separable locally compact group consisting of an increasing sequence of compact groups of $*$-automorphisms of a C^*-algebra $\mathcal{A}$ φ is a G-strongly quasi-invariant state on $\mathcal{A}$. We have a von Neumann algebra $\mathfrak{A} = \pi(\mathcal{A})''$ of the GNS representation of $(\mathcal{A}, \varphi)$. We assume the Hypothesis (H). Therefore, $\mathfrak{A}$ is equipped with a $\widehat{G}$-invariant state $\psi_G(\cdot) = \frac{1}{\|\Phi_G\|^2}\langle \Phi_G, \cdot \Phi_G\rangle$. For each N, define

$$E_N(x) := \int_{G_N} u_g(x)dg, \quad x \in \mathfrak{A}. \tag{4.1}$$

By Theorem 2.5, E_N is an Umegaki conditional expectation onto $\mathfrak{B}_N := \mathrm{Fix}(u_{G_N})$. In particular, the following relations hold:

$$E_N(\pi(a)) = \int_{G_N} u_g(\pi(a))dg = \int_{G_N} \pi(g(a))dg, \quad \forall a \in \mathcal{A}, \tag{4.2}$$

$$u_h E_N(\pi(a)) = \int_{G_N} \pi(hg(a))dg = E_N(\pi(a)). \tag{4.3}$$

Obviously, $(\mathfrak{B}_N)_N$ is a decreasing sequence of von Neumann subalgebras of $\mathfrak{A}$.

Theorem 4.1 *Suppose the Hypothesis (H). It holds that $\psi_G \circ E_N = \psi_G$ for all N and the sequence $(E_N)_N$ satisfies the martingale property, namely, for $M \le N$,*

$$E_N E_M = E_M E_N = E_N. \tag{4.4}$$

Proof. By Theorem 3.2, the state $\psi_G = \frac{1}{\|\Phi_G\|^2}\langle \Phi_G, \cdot \Phi_G\rangle$ is u_g-invariant for all $g \in G$. Therefore, for all $x \in \mathfrak{A}$ and $N \in \mathbb{N}$,

$$\begin{aligned} \psi_G(E_N(x)) &= \psi_G\left(\int_{G_N} u_g(x)dg\right) \\ &= \int_{G_N} \psi_G(u_g(x))dg \\ &= \int_{G_N} \psi_G(x)dg \\ &= \psi_G(x). \end{aligned}$$

Now suppose that $M \le N$. One has

$$\begin{aligned} E_N E_M(\pi(a)) &= E_N\left(\int_{G_M} \pi(h(a))dh\right) \\ &= \int_{G_M}\left(\int_{G_N} \pi(gh(a))dg\right)dh \\ &= \int_{G_M}\left(\int_{G_N} \pi(g(a))dg\right)dh \\ &= \int_{G_M} E_N(\pi(a))\, dh = E_N(\pi(a)). \end{aligned}$$

The relation $E_M E_N = E_N$ can be similarly shown and the proof is completed. □

Theorem 4.2 *Suppose that the Hypothesis (H) is satisfied and the state ψ_G is faithful. Then, for any $x \in \mathfrak{A}$, the sequence $(x_N)_N$, where $x_N := E_N(x)$, is a (backward) martingale, and hence has a strong limit. Furthermore, by defining $E_\infty(x) := \lim_N E_N(x)$, E_∞ is an Umegaki conditional expectation onto* $\mathrm{Fix}(u_G)$.

Proof. The martingale property has been shown in Theorem 4.1, i.e., for $M \le N$, $E_N(x_M) = x_N$. The convergence of $(x_N)_N$, namely a (backward) martingale convergence theorem is well known (Ref. 11, Theorem 4). Now let us define

$$E_\infty(x) := \lim_N E_N(x), \quad x \in \mathfrak{A}. \tag{4.5}$$

Obviously, $\mathrm{Ran}E_\infty \subset \cap_N \mathrm{Fix}(u_{G_N}) = \mathrm{Fix}(u_G)$. On the other hand, suppose that $x \in \mathrm{Fix}(u_G)$. Then, for each $N \in \mathbb{N}$, $x \in \mathrm{Fix}(u_{G_N})$ and $E_N(x) = x$. Thus $E_\infty(x) = \lim_N E_N(x) = x$ showing that $E_\infty : \mathfrak{A} \to \mathrm{Fix}(u_G)$ is an onto map. Furthermore, for any $y, y' \in \mathrm{Fix}(u_G)$ and $x \in \mathfrak{A}$, we have

$$E_\infty(yxy') = \lim_N E_N(yxy') \overset{y,y' \in \mathrm{Fix}(u_{G_N})}{=} \lim_N (yE_N(x)y') = yE_\infty(x)y'.$$

And, for any $x \in \mathfrak{A}$,

$$\psi_G \circ E_\infty(x) = \psi_G(\lim_N E_N(x)) = \lim_N \psi_G \circ E_N(x) \overset{\psi_G \circ E_N = \psi_G}{=} \psi_G(x).$$

Therefore, $E_\infty : (\mathfrak{A}, \psi_G) \to \mathfrak{B}_\infty \equiv \mathrm{Fix}(u_G)$ is an Umegaki conditional expectation. □

V. EXAMPLE: THE GROUP OF LOCAL PERMUTATIONS

Let $\mathbb{N}_0 = \{0, 1, 2, \cdots\}$ be the set of nonnegative integers and let $\mathcal{S}_\infty = \cup_{N \in \mathbb{N}_0} \mathcal{S}_N$ be the group of local permutations on $\mathbb{N}_0$. Putting $\mathcal{B} := \mathcal{B}(\mathbb{C}^d)$, let $\mathcal{A} = \otimes_{n \in \mathbb{N}_0} \mathcal{B}$ be the C^*-algebra of infinite direct product of copies $\mathcal{B}$.[13] To say more precisely, for each finite set $F \subset \mathbb{N}_0$, let $\mathcal{A}_F := \otimes_{n \in F} \mathcal{B}$ be the C^*-algebra of finite direct product of $\mathcal{B}$'s. For $F_1 \subset F_2$, there is a natural embedding $\mathcal{A}_{F_1} \subset \mathcal{A}_{F_2}$ and $\mathcal{A} = \otimes_{n \in \mathbb{N}_0} \mathcal{B}$ is defined as the C^*-inductive limit of $\mathcal{A}_F$'s. We may consider $\mathcal{A}_F$ as a C^*-subalgebra of $\mathcal{A}$ and in particular for each $n \in \mathbb{N}_0$, $\mathcal{A}_n \equiv j_n(B) := (\otimes_{k \le n-1} \mathbb{1}_{\mathcal{B}}) \otimes \mathcal{B} \otimes (\otimes_{k \ge n+1} \mathbb{1}_{\mathcal{B}})$ is a subalgebra obtained by embedding $\mathcal{B}$ into the nth position.

For each $n \in \mathbb{N}_0$, let $W_n \in \mathcal{B}$ be a density matrix and let $\varphi(\cdot) := \otimes_{n \in \mathbb{N}_0} \mathrm{tr}(W_n \cdot)$ be a product state on $\mathcal{A}$. We assume $(W_n, W_m) = 0$ for all $m, n \in \mathbb{N}_0$. In a moment we will see that φ is G-strongly quasi invariant for $G = \mathcal{S}_\infty$, but here we remark that we are considering the finite dimensional algebra $\mathcal{B}(\mathbb{C}^d)$, otherwise we need to consider the generalized G-strongly quasi invariance in the sense that the cocycles x_g are allowed to be unbounded (Ref. 6, Definition 2.2). Let $(b_n)_n$ be a sequence of element of $\mathcal{B}$ such that $b_n = \mathbb{1}$ except finitely many n's. Then

$$\varphi\left(\prod_{n \in \mathbb{N}_0} j_n(b_n)\right) = \prod_{n \in \mathbb{N}_0} \varphi_n(b_n) = \mathrm{tr}_{\mathbb{N}_0}\left(\prod_{n \in \mathbb{N}_0} j_n(W_n) \prod_{n \in \mathbb{N}_0} j_n(b_n)\right)$$

where $\varphi_n(b_n) = \mathrm{tr}(W_n b_n)$ is a state on $\mathcal{B}$ and $\mathrm{tr}_{\mathbb{N}_0}(\cdot) = \otimes_{\mathbb{N}_0} \mathrm{tr}(\cdot)$. Then for any $\sigma \in \mathcal{S}_N$, one has

$$\begin{aligned}
\varphi\left(\sigma\left(\prod_n j_n(b_n)\right)\right) &= \mathrm{tr}_{\mathbb{N}_0}\left(\prod_n j_n(W_n) \sigma\left(\prod_n j_n(b_n)\right)\right) \\
&= \mathrm{tr}_{\mathbb{N}_0}\left(\sigma^{-1}\left(\prod_n j_n(W_n)\right) \prod_n j_n(b_n)\right) \\
&= \mathrm{tr}_{\mathbb{N}_0}\left(\prod_n j_n(W_{\sigma(n)})\right) \prod_n j_n(b_n)\Big) \\
&= \mathrm{tr}_{\mathbb{N}_0}\left(\prod_n j_n(W_n) \prod_n j_n(W_n{}^{-1} W_{\sigma(n)}) \prod_n j_n(b_n)\right).
\end{aligned}$$

Therefore, one gets

$$x_\sigma = \prod_{n \in \Lambda_\sigma} j_n(W_n{}^{-1} W_{\sigma(n)}), \tag{5.1}$$

where Λ_σ is the support of σ meaning that $\sigma(j) = j$ for $j \notin \Lambda_\sigma$.

Let $W \in \mathcal{B}$ be a fixed density matrix and let F be a finite subset of $\mathbb{N}_0$. We assume that

$$W_n = W, \quad \forall n \in F^c. \tag{5.2}$$

We also assume that there exists a constant $C > 1$ such that

$$\frac{1}{C} \le \|W_n\| \le C, \quad n \in \mathbb{N}_0. \tag{5.3}$$

The Hypothesis (H) was crucial in this paper. We first check that (H) holds in this model.

Lemma 5.1 The Hypothesis (H) holds for the above model.

Proof. For the proof we will use Proposition 3.4. Fix an $m_0 \in \mathbb{N}_0$ such that $F \subset \{0, 1, \ldots, m_0\}$. For $N > m_0$, define (see Fig. 1)

$$\mathcal{S}_N^{(m_0)} = \{\sigma \in \mathcal{S}_N : \ \sigma(k) > m_0, \ \ \sigma^{-1}(k) > m_0 \quad \text{if } k \in \{0, 1, \ldots, m_0\}\}, \tag{5.4}$$

and we put $A_N := \mathcal{S}_N^{(m_0)}$. Denoting by $|A|$ the cardinality of a set A, it is not hard to see that

$$\lim_{N \to \infty} \frac{1}{N!} |\mathcal{S}_N \setminus \mathcal{S}_N^{(m_0)}| = 0. \tag{5.5}$$

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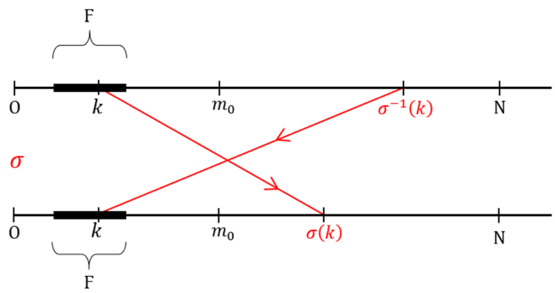

FIG. 1. $\sigma \in \mathcal{S}_N^{(m_0)}$.

Therefore, we can find a $\delta_0 > 0$ and $N_0 > m_0$ such that for all $N \geq N_0$, $\lambda_N(A_N) \geq \delta_0$. Now, for $N \geq N_0$, if $\sigma \in A_N$, by (5.1) we see that

$$x_\sigma = \prod_{n \in \Lambda_\sigma} j_n(W_n{}^{-1} W_{\sigma(n)}) = \prod_{n \in F} j_n(W_n{}^{-1} W) j_{\sigma^{-1}(n)}(W^{-1} W_n).$$

Therefore, for $\sigma, \tau \in A_N$ we compute

$$\varphi((x_\sigma x_\tau)^{1/2}) = \prod_{\substack{n \in F: \\ \sigma^{-1}(n) \neq \tau^{-1}(n)}} \mathrm{tr}\Big(W^{1/2}_{\sigma^{-1}(n)} W^{1/2}\Big) \mathrm{tr}\Big(W^{1/2}_{\tau^{-1}(n)} W^{1/2}\Big) \geq C^{-2N(\sigma,\tau)} \geq C^{-2|F|} > 0,$$

where $N(\sigma, \tau) = |\{n \in F : \sigma^{-1}(n) \neq \tau^{-1}(n)\}|$. Thus, the conditions of Proposition 3.4 are fulfilled and we are done. □

In the next proposition we show a stronger result, namely, the weak convergence of (K_N). It is another support for the Hypothesis (H) by Proposition 3.5.

Proposition 5.2 Under the assumptions (5.2) and (5.3), *the sequence* $(K_N)_N$ *converges weakly to an operator* K_G *given by*

$$K_G = \Bigg(\prod_{n \in F} \mathrm{tr}(W_n^{1/2} W^{1/2})\Bigg) \pi\Bigg(\prod_{n \in F} j_n\Big((W_n{}^{-1} W)^{1/2}\Big)\Bigg). \tag{5.6}$$

Proof. Fix a number $n_0 \in \mathbb{N}_0$ such that $F \subset \{0, 1, \ldots, n_0\}$. Let $a = \prod_{n \in \mathbb{N}_0} j_n(a_n)$ and $b = \prod_{n \in \mathbb{N}_0} j_n(b_n)$ be the elements of $\mathcal{A}$ such that there exists a $k_0 \in \mathbb{N}_0$ with $a_n = b_n = \mathbb{1}$ for $n \geq k_0$. For big enough N's such that $N > m_0 := \max\{n_0, k_0\}$ we decompose $\mathcal{S}_N$ as

$$\mathcal{S}_N = \mathcal{S}_N^{(m_0)} \cup (\mathcal{S}_N \backslash \mathcal{S}_N^{(m_0)}),$$

where $\mathcal{S}_N^{(m_0)}$ is defined in (5.8). In order to show the weak convergence of $(K_N)_N$, let us compute the limit of $\langle \pi(a)\Phi, K_N \pi(b)\Phi\rangle$. From (5.1), we have (we will omit the representation symbol "π" whenever there is no danger of confusion)

$$K_N = \frac{1}{N!} \sum_{\sigma \in \mathcal{S}_N} \prod_{n \in \Lambda_\sigma} j_n(W_n^{-1/2} W^{1/2}_{\sigma(n)}).$$

Therefore,

$$\begin{aligned}
&\langle \pi(a)\Phi, K_N \pi(b)\Phi\rangle \\
&= \frac{1}{N!} \sum_{\sigma \in \mathcal{S}_N} \mathrm{tr}_{\mathbb{N}_0}\Bigg(\Bigg(\prod_{n \in \mathbb{N}_0} j_n(W_n)\Bigg)\Bigg(\prod_{n \in \mathbb{N}_0} j_n(a_n^*)\Bigg)\Bigg(\prod_{n \in \Lambda_\sigma} j_n(W_n^{-1/2} W^{1/2}_{\sigma(n)})\Bigg)\Bigg(\prod_{n \in \mathbb{N}_0} j_n(b_n)\Bigg)\Bigg) \\
&= \frac{1}{N!}\Bigg(\sum_{\sigma \in \mathcal{S}_N^{(m_0)}} + \sum_{\sigma \in \mathcal{S}_N \backslash \mathcal{S}_N^{(m_0)}}\Bigg) \mathrm{tr}_{\mathbb{N}_0}\Bigg(\Bigg(\prod_{n \in \mathbb{N}_0} j_n(W_n)\Bigg)\Bigg(\prod_{n \in \mathbb{N}_0} j_n(a_n^*)\Bigg)\Bigg(\prod_{n \in \Lambda_\sigma} j_n(W_n^{-1/2} W^{1/2}_{\sigma(n)})\Bigg)\Bigg(\prod_{n \in \mathbb{N}_0} j_n(b_n)\Bigg)\Bigg) \\
&=: (\mathrm{outer})_N + (\mathrm{inner})_N.
\end{aligned}$$

By using (5.5), it can be easily shown that

$$|(\text{inner})_N| \le \frac{1}{N!}|\mathcal{S}_N \setminus \mathcal{S}_N^{(m_0)}| \cdot C^{4|F|} M^{2k_0} \xrightarrow[N\to\infty]{} 0, \tag{5.7}$$

where $M := \max\{\|a_n\|, \ \|b_n\| : \ n \in \mathbb{N}_0\}$. In order to estimate the first term, suppose that $\sigma \in \mathcal{S}_N^{(m_0)}$. Then, one sees that (see Fig. 1)

$$\begin{aligned}\prod_{n\in\Lambda_\sigma} j_n(W_{\sigma(n)}^{1/2}/W_n^{1/2}) &= \left(\prod_{k\in F} j_k(W_k^{-1/2}W_{\sigma(k)}^{1/2})\right)\left(\prod_{k\in F} j_{\sigma^{-1}(k)}(W_{\sigma^{-1}(k)}^{-1/2}W_k^{1/2})\right)\\ &= \left(\prod_{k\in F} j_k(W_k^{-1/2}W^{1/2})\right)\left(\prod_{k\in F} j_{\sigma^{-1}(k)}(W^{-1/2}W_k^{1/2})\right).\end{aligned}$$

Also for those σ, we see that $a_k = \mathbb{1}$ and $b_k = \mathbb{1}$ for $k \in \sigma^{-1}(F)$. Therefore, if $\sigma \in \mathcal{S}_N^{(m_0)}$, then

$$\begin{aligned}&\mathrm{tr}_{\mathbb{N}_0}\left(\left(\prod_{n\in\mathbb{N}_0} j_n(W_n)\right)\left(\prod_{n\in\mathbb{N}_0} j_n(a_n^*)\right)\left(\prod_{n\in\Lambda_\sigma} j_n(W_n^{-1/2}W_{\sigma(n)}^{1/2})\right)\left(\prod_{n\in\mathbb{N}_0} j_n(b_n)\right)\right)\\ &= \mathrm{tr}_{\mathbb{N}_0}\left(\left(\prod_{n\in\mathbb{N}_0} j_n(W_n)\right)\left(\prod_{n\in\mathbb{N}_0} j_n(a_n^*)\right)K_G\left(\prod_{n\in\mathbb{N}_0} j_n(b_n)\right)\right)\\ &= \langle \pi(a)\Phi, K_G\pi(b)\Phi\rangle,\end{aligned}$$

where K_G is defined in (5.6). Now since $\frac{1}{N!}|\mathcal{S}_N^{(m_0)}| \xrightarrow[N\to\infty]{} 1$, we have

$$\lim_{N\to\infty} (\text{outer})_N = \langle \pi(a)\Phi, K_G\pi(b)\Phi\rangle. \tag{5.8}$$

Combining (5.7) and (5.8) we see that K_n converges weakly to K_G. □

Remark 5.3 Let us compute the G-invariant state φ_G in Corollary 3.3 for the example in this section. Let $a = \otimes_{n\in\mathbb{N}_0} a_n$ be any element of the form in the beginning of the Proof of Proposition 5.2. By noticing $\|\Phi_G\|^2 = \|K_G\Phi\|^2 = \left(\prod_{n\in F} \mathrm{tr}(W_n^{1/2}W^{1/2})\right)^2$,

$$\begin{aligned}\varphi_G(a) &= \psi_G(\pi(a))\\ &= \frac{1}{\|\Phi_G\|^2}\langle \Phi_G, \pi(a)\Phi_G\rangle\\ &= \frac{1}{\|K_G\Phi\|^2}\langle K_G\Phi, \pi(a)K_G\Phi\rangle\\ &= \mathrm{tr}_{\mathbb{N}_0}\left(\left(\prod_{n\in\mathbb{N}_0} j_n(W_n)\right)\left(\prod_{n\in F} j_n((W_n{}^{-1}W)^{1/2})\right)\left(\prod_{n\in\mathbb{N}_0} j_n(a_n)\right)\left(\prod_{n\in F} j_n((W_n{}^{-1}W)^{1/2})\right)\right)\\ &= \mathrm{tr}_{\mathbb{N}_0}\left(\left(\prod_{n\in F} j_n(W_n)\right)\left(\prod_{n\in F^c} j_n(W)\right)\left(\prod_{n\in F} j_n((W_n{}^{-1}W)^{1/2})\right)^2\left(\prod_{n\in\mathbb{N}_0} j_n(a_n)\right)\right)\\ &= \mathrm{tr}_{\mathbb{N}_0}\left(\left(\prod_{n\in\mathbb{N}_0} j_n(W)\right)\left(\prod_{n\in\mathbb{N}_0} j_n(a_n)\right)\right)\\ &= \prod_{n\in\mathbb{N}_0} \mathrm{tr}(Wa_n).\end{aligned}$$

Obviously, φ_G is G-invariant and is the infinite product of the identical states: $\varphi_G = \otimes_{n\in\mathbb{N}_0}\varphi_n$, $\varphi_n(\cdot) = \mathrm{tr}(W\cdot)$.

VI. AN APPLICATION TO CLASSICAL SPIN SYSTEMS

In this section we consider the classical spin systems in the statistical mechanical models.

A. Gibbs measures

Let $\mathbb{Z}^d$ be the d-dimensional integer space. Let $\Omega_0 = \{1, -1\}$ be the set representing the spins at each site. The set Ω_0 is equipped with a Bernoulli distribution μ_0: $\mu_0(\{1\}) = \mu_0(\{-1\}) = \frac{1}{2}$. We let μ_Λ be the probability measure μ_0^Λ on the set $\Omega_\Lambda := \Omega_0^\Lambda$. The whole configuration space is denoted by $\Omega := \Omega_{\mathbb{Z}^d}$. For each $\Lambda \subset \mathbb{Z}^d$, $\mathcal{F}_\Lambda$ denotes a σ-algebra on the set Ω generated by the spin variables in the set Λ. We simply denote by $\mathcal{F}$ for $\mathcal{F}_{\mathbb{Z}^d}$. In the sequel, when $\Lambda \subset \mathbb{Z}^d$ is a finite subset we denote it by $\Lambda \subset\subset \mathbb{Z}^d$.

We abuse the notations but following the tradition of statistical mechanics, by an interaction $\Phi = (\Phi_\Lambda)_{\Lambda\subset\subset\mathbb{Z}^d}$ we mean a set of real-valued functions $\Phi_\Lambda : \Omega \to \mathbb{R}$, which is $\mathcal{F}_\Lambda$-measurable.

Example 6.1 **(Ising model).** The interaction for Ising model with a nearest neighborhood interaction is given by for $\xi = (\xi_i)_{i\in\mathbb{Z}^d} \in \Omega$

$$\Phi_\Lambda(\xi) = \begin{cases} -J\xi_i\xi_j, & \Lambda = \{i, j\}, \quad |i - j| = 1, \\ h\xi_i, & \Lambda = \{i\}, \\ 0, & \text{otherwise.} \end{cases}$$

Here J is the interaction strength and h denotes the external magnetic field strength; $J > 0$ for ferro magnetic model and $J < 0$ for anti-ferro magnetic model.

In the sequel we assume that the interaction is translation invariant, i.e., $\Phi_X(\sigma_i(\omega)) = \Phi_{\sigma_i^{-1}(X)}(\omega)$, where σ_i is the translation by i, i.e., for all $j \in \mathbb{Z}^d$, $\sigma_i(j) = j + i$ and $(\sigma_i(\omega))_j = \omega_{j+i}$ for $\omega \in \Omega$. Furthermore, we assume that

$$\sum_{X\ni 0} \|\Phi_X\|_\infty < \infty. \tag{6.1}$$

Given an interaction Φ, for each $\Lambda \subset\subset \mathbb{Z}^d$ define a function $H_\Lambda(\cdot|\cdot)$, the Hamiltonian with boundary condition, by

$$H_\Lambda^\Phi(\zeta|\omega) := \sum_{X\cap\Lambda\neq\emptyset} \Phi_X(\zeta_\Lambda\omega_{\Lambda^c}), \quad \zeta, \quad \omega \in \Omega, \tag{6.2}$$

where $\zeta_\Lambda\omega_{\Lambda^c} \in \Omega$ is the juxtaposition of ζ_Λ and ω_{Λ^c}, which are the restrictions of ζ and ω on Λ and Λ^c, respectively. Let us define

$$Z_\Lambda^\Phi(\omega) := \int_{\Omega_\Lambda} \exp\left[-H_\Lambda^\Phi(\zeta|\omega)\right] d\mu_\Lambda(\zeta_\Lambda).$$

We give a definition of a Gibbs measure for the interaction Φ [Ref. 8, Definition (2.9)].

Definition 6.2 Let Φ be an interaction satisfying (6.1) and $\omega \in \Omega$, $\Lambda \subset\subset \mathbb{Z}^d$. Then the probability measure

$$A \mapsto \gamma_\Lambda^\Phi(A|\omega) := \frac{1}{Z_\Lambda^\Phi(\omega)} \int_{\Omega_\Lambda} \exp\left[-H_\Lambda^\Phi(\zeta|\omega)\right] 1_A(\zeta_\Lambda\omega_{\Lambda^c}) d\mu_\Lambda(\zeta_\Lambda)$$

on $(\Omega, \mathcal{F})$ is called the Gibbs distribution in Λ with boundary condition ω and interaction Φ. The system $(\gamma_\Lambda^\Phi)_{\Lambda\subset\subset\mathbb{Z}^d}$ is called the Gibbsian specification for Φ. Any probability measure μ on $(\Omega, \mathcal{F})$ is called a Gibbs measure for the interaction Φ if it satisfies the so called Dobrushin–Lanford–Ruelle (DLR) equations:

$$\mathbb{E}_\mu[A|\mathcal{F}_{\Lambda^c}] = \gamma_\Lambda^\Phi(A|\cdot) \quad \mu - \text{a.s.} \quad \text{for all } A \in \mathcal{F} \text{ and } \Lambda \subset\subset \mathbb{Z}^d. \tag{6.3}$$

The set of Gibbs measures for the interaction Φ is denoted by $\mathcal{G}(\Phi)$.

Remark 6.3

(i) It is known that under the condition (6.1). $\mathcal{G}(\Phi) \neq \emptyset$.[8,12]
(ii) Let Ω be equipped with the product topology considering Ω_0 as a discrete topological space. For any continuous function $f \in C(\Omega)$ on Ω, define

$$\gamma_\Lambda^\Phi(f|\omega) := \int_\Omega f(\zeta_\Lambda\omega_{\Lambda^c})\gamma_\Lambda^\Phi(d\zeta|\omega) = \frac{1}{Z_\Lambda^\Phi(\omega)} \int_{\Omega_\Lambda} \exp\left[-H_\Lambda^\Phi(\zeta|\omega)\right] f(\zeta_\Lambda\omega_{\Lambda^c}) d\mu_\Lambda(\zeta_\Lambda).$$

The DLR equation (6.3) is equivalent to saying that

$$\mu(f) := \int f(\omega) d\mu(\omega) = \mu(\gamma_\Lambda^\Phi(f|\cdot)), \quad f \in C(\Omega), \quad \Lambda \subset\subset \mathbb{Z}^d. \tag{6.4}$$

B. Group actions and strong quasi-invariance of Gibbs measures

From now on we assume that the interaction is of finite range, i.e., there is an $R > 0$ such that $\Phi_X = 0$ if $\operatorname{diam}(X) > R$.

Notice that Ω is a compact space and let $\mathcal{A}$ be the C^*-algebra $C(\Omega)$ equipped with the sup-norm. Any probability measure on $(\Omega, \mathcal{F})$, in particular any Gibbs measure, is a state on $\mathcal{A}$. For each $N \in \mathbb{N}$, let $\Lambda_N = [-N, N]^d \cap \mathbb{Z}^d$ denote the hypercube of side length $2N + 1$. Let $(G_N)_{N\in\mathbb{N}}$ be an increasing sequence of automorphisms of $\mathcal{A}$ such that for each $N \in \mathbb{N}$, G_N depends only on the local configurations in Λ_N. We let $G = \cup_{N\in\mathbb{N}} G_N$. For the group G, mostly we have in mind the group of spin interchanges or spin flips defined as follows:

Example 6.4 We consider the following group of automorphisms. Notice that any continuous bijection $\tau: \Omega \to \Omega$ naturally induces an automorphism $\tau: \mathcal{A} \to \mathcal{A}$ by

$$\tau(f)(\omega) = f(\tau(\omega)), \quad f \in \mathcal{A}.$$

(i) (Spin exchanges) For $i \neq j \in \mathbb{Z}^d$, $\tau_{ij}: \Omega \to \Omega$ is defined by

$$(\tau_{ij}(\omega))_k = \omega_k^{ij} := \begin{cases} \omega_k, & k \neq i, j \\ \omega_j, & k = i, \\ \omega_i, & k = j. \end{cases}.$$

The group G_N is generated by $\{\tau_{ij} : i \neq j \in \Lambda_N\}$. In other words, G_N consists of spin permutations in the cube Λ_N.

(ii) (Spin flips) For each $i \in \mathbb{Z}^d$, $\tau_i: \Omega \to \Omega$ is defined by

$$(\tau_i(\omega))_j = \omega_j^i := \begin{cases} \omega_j, & j \neq i, \\ -\omega_i, & j = i. \end{cases}.$$

The group G_N is generated by $\{\tau_i : i \in \Lambda_N\}$ so it is the group of partial spin flips in Λ_N.

Theorem 6.5 *Let Φ be an interaction satisfying (6.1) and let μ be a Gibbs measure for Φ. Let $G = \cup_{N\in\mathbb{N}} G_N$ be one of the locally compact groups introduced in Example 6.4. Then μ is G-strongly quasi-invariant with cocycles x_τ given by*

$$x_\tau(\omega) = \exp[H(\omega) - H(\tau^{-1}(\omega))], \quad \tau \in G, \tag{6.5}$$

here, and in the sequel, the exponent is defined by

$$H(\omega) - H(\tau^{-1}(\omega)) = \lim_{N\to\infty} \sum_{X \subset \Lambda_N} \left(\Phi_X(\omega) - \Phi_X(\tau^{-1}(\omega))\right)],$$

which is well-defined since τ gives only a local change.

Proof. Suppose that $\tau \in G_{N_0}$. Let $N > N_0 + R$ and fix $\Lambda \subset\subset \mathbb{Z}^d$ such that $\Lambda_N \subset \Lambda$. By (6.4), we have

$$\begin{aligned}
\mu(\tau(f)) &= \mu(\gamma_{\Lambda_N}^\Phi(\tau(f)|\cdot)) \\
&= \int \left(\frac{1}{Z_\Lambda^\Phi(\omega)} \int_{\Omega_\Lambda} \exp\left[-H_\Lambda^\Phi(\zeta|\omega)\right] \tau(f)(\zeta_\Lambda \omega_{\Lambda^c}) d\mu_\Lambda(\zeta_\Lambda) \right) d\mu(\omega) \\
&= \int \left(\frac{1}{Z_\Lambda^\Phi(\omega)} \int_{\Omega_\Lambda} \exp\left[-H_\Lambda^\Phi(\zeta|\omega)\right] f(\tau(\zeta_\Lambda)\omega_{\Lambda^c}) d\mu_\Lambda(\zeta_\Lambda) \right) d\mu(\omega) \\
&= \int \left(\frac{1}{Z_\Lambda^\Phi(\omega)} \int_{\Omega_\Lambda} \exp\left[-H_\Lambda^\Phi(\tau^{-1}(\zeta)|\omega)\right] f(\zeta_\Lambda \omega_{\Lambda^c}) d\mu_\Lambda(\zeta_\Lambda) \right) d\mu(\omega)
\end{aligned}$$

$$= \int \left(\frac{1}{Z_\Lambda^\Phi(\omega)} \int_{\Omega_\Lambda} \exp\left[-H_\Lambda^\Phi(\zeta|\omega)\right](x_\tau f)(\zeta_\Lambda \omega_{\Lambda^c}) d\mu_\Lambda(\zeta_\Lambda) \right) d\mu(\omega),$$

where

$$x_\tau(\zeta) = \exp\left[\sum_{X\cap\Lambda_{N_0}\neq\emptyset} \left(\Phi_X(\zeta) - \Phi_X(\tau^{-1}(\zeta))\right)\right]$$
$$= \exp\left[H(\zeta) - H(\tau^{-1}(\zeta))\right].$$

□

Let $(\mathcal{H}_\mu, \pi_\mu, \Phi_\mu)$ be the GNS representation of $(\mathcal{A}, \mu)$. We can think of $\mathcal{H}_\mu = L^2(\Omega, \mu)$, $\pi_\mu(f) = f$ for $f \in C(\Omega)$, $\mathfrak{A} = \pi_\mu(\mathcal{A})'' = L^\infty(\Omega, \mu)$ acting as multiplication operators on $L^2(\Omega, \mu)$, and $\Phi_\mu = 1$, the unit function on Ω. Let us compute the unitary operators U_τ, $\tau \in G_N$, in (2.1).

Given an $f \in C(\Omega)$ and $\tau \in G$, recall from (2.1) that

$$U_\tau \pi_\mu(f)\Phi_\mu = \pi_\mu(\tau(f) x_{\tau^{-1}}^{1/2})\Phi_\mu, \tag{6.6}$$

and by (6.5)

$$(\tau(f) x_{\tau^{-1}}^{1/2})(\omega) = f(\tau(\omega)) \exp\left[\frac{1}{2}(H(\omega) - H(\tau(\omega)))\right]. \tag{6.7}$$

We also recall for each $\tau \in G$ an automorphism on $\mathfrak{A}$:

$$u_\tau \pi_\mu(f) = U_\tau \pi_\mu(f) U_\tau^*.$$

By the results of subsection 2.2 we have a sequence of conditional expectations $(E_N)_N$:

$$E_N(\pi_\mu(f)) = \int_{G_N} u_\tau(\pi_\mu(f)) d\tau.$$

Now let us consider the projections $P_N = \int_{G_N} U_\tau d\tau$ and the vector $\Phi_N = P_N \Phi_\mu$. By (6.6) and (6.7)

$$\Phi_N(\omega) = \int_{G_N} (U_\tau \Phi_\mu)(\omega) d\tau$$
$$= \frac{1}{|G_N|} \sum_{\tau \in S_{\Lambda_N}} \exp\left[\frac{1}{2}(H(\omega) - H(\tau(\omega)))\right]. \tag{6.8}$$

Defining unit vectors on $\mathcal{H}_\mu$ by

$$\Psi_N := \frac{1}{\|\Phi_N\|} \Phi_N,$$

we get vector states ψ_N on $\mathfrak{A}$ by

$$\psi_N(f) = \langle \Psi_N, f \Psi_N \rangle_{\mathcal{H}_\mu}, \quad f \in \mathfrak{A}. \tag{6.9}$$

By Banach–Alaoglu theorem there is a subnet $(\psi_{N_k})_k$ such that it converges in weak*-topology to a state, say ψ, on $\mathfrak{A}$. It is obvious that ψ is G-invariant. We have the following result analogous to Theorem 4.2.

Theorem 6.6 *We assume that the state ψ is faithful. Then the state ψ is $\widehat{G}$-invariant, i.e., $\psi(u_\tau(\pi_\mu(f))) = \psi(\pi_\mu(f))$ for all $f \in C(\Omega)$, and for any element $x \in \mathfrak{A}$, the sequence $(x_N)_N$, $x_N = E_N(x)$, is a backward martingale and has a limit $E_\infty(x) := \lim_N E_N(x)$.*

Proof. The proof is exactly the same as in Theorem 4.2. □

Remark 6.7 We were not able to show that the limit $\Phi_G := \lim_N \Phi_N$ is non-zero. However, for the ferromagnetic Ising model, at least in the low-temperature regime, it must be true. The intuitive reasoning is as follows. Suppose the extreme case of zero temperature. Then the Gibbs measure has support on the configuration of all +1-spins or on the configuration of all −1-spins. In this case, obviously $x_\tau = 1$ for all $\tau \in G$ (of course μ is G-invariant). Now if the temperature is not zero but sufficiently low, then there is a big cluster of same spins. In that case the exponent in the computation of x_τ in (6.5) is close to zero for many τ's, meaning that x_τ is close to 1. So, the conditions of Proposition 3.4 are satisfied.

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

Ameur Dhahri: Methodology (equal); Writing – original draft (equal). **Chul Ki Ko**: Methodology (equal); Writing – original draft (equal). **Hyun Jae Yoo**: Methodology (equal); Writing – original draft (equal).

DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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