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# Entropy production of quantum Markov semigroup associated with open quantum walks on the periodic graphs

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**Abstract**
In this paper, we compute the entropy production of quantum Markov semigroup associated with open quantum walks. The entropy production, for the classical as well as quantum systems, measures the deviation from the symmetry between the forward and backward processes. The detailed balance condition with respect to an invariant state is the condition for the symmetry of the dynamics. Here we consider the quantum Markov semigroups associated with open quantum walks on the periodic graphs. On the one hand, the model serves as a good example to study the quantum detailed balance condition and the entropy production. On the other hand, from the viewpoint of the dynamics itself, the concept of entropy production helps for a better understanding of the dynamics.

**Keywords** Entropy production · Quantum detailed balance condition · Open quantum walks

**Mathematics Subject Classification** 81P17 · 81R15 · 82C41

## 1 Introduction

In this paper we, discuss the entropy production for a quantum Markov semigroup with a stationary state associated with open quantum walks on the periodic graphs. The aim is twofold: one is to find an interesting model to explain the concept of

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quantum entropy production. The other one is to show that the concept of entropy production is a good tool for a better understanding of the dynamics.

For the classical interacting particle systems [19], among others, in a series of papers Maes et al. developed the concept of entropy production [22–24, and references therein]. There the authors compare the steady state with the state of the time-reversed process. The relative entropy resulting from this comparison defines the entropy production. The Gibbsian formalism under consideration and the Gibbs measures as steady states define not only the dynamics itself but also the detailed balance condition via the interaction potentials.

Analogous to the theory applied to classical Markov semigroups for interacting particle systems, Fagnola and Rebolledo introduced a definition of entropy production rate for faithful normal invariant states of quantum Markov semigroups [8–10]. In [9], the entropy production is defined as the relative entropy of the one-step forward and backward two-point states (see Sect. 3 for the details). The entropy production for quantum Markov semigroups, as for the classical stochastic processes, measures a deviation from detailed balance (see [9, and references therein]). The computation of entropy production for QMS in some models can be found for instance in [5, 9, 10].

The symmetric Markov semigroup plays a central role in the classical stochastic dynamical systems [14, 21] providing a deep connection of analytic and probabilistic methods in the study of the dynamics. When the Markov semigroup has an invariant state (a probability measure), the detailed balance condition gives a sufficient condition for the symmetry [19]. The extension of the concept of symmetry and detailed balance condition to the quantum dynamical systems have been established in several ways by many authors (see for example [1–4, 6, 11–13, 15, 18, 26, 27] and references therein). In this paper, we follow the definitions developed by Fagnola and Umanita in [11–13]. In particular, they characterized the KMS symmetry and quantum detailed balance conditions by the Gorini–Kossakowski–Sudarshan [16] and Lindblad [20] (GKSL) generators (see Sect. 2 for the details).

In this paper, we will concretely compute the entropy production for the quantum Markov semigroups associated with open quantum walks with periodic boundary conditions. Particularly we will consider the model on the cycle and torus from the regular integer lattices and also the model on the crystal torus (Sects. 4 and 5). We also investigate the relationship between no-existence of entropy production and quantum detailed balance condition (Sect. 6).

The paper is organized as follows. In Sect. 2 we introduce the GKSL generators for the quantum Markov semigroups and then the KMS symmetry and quantum detailed balance conditions. In Sect. 3, we briefly review the entropy production formula for quantum Markov semigroups following [8–10]. In the following Sects. 4 and 5, we compute the entropy productions in our model. In the final Sect. 6, we discuss the relationship between the quantum detailed balance condition and no-existence of entropy production in the considered model.

## 2 Quantum Markov semigroups, detailed balance condition and KMS symmetry

In this paper we consider the norm continuous quantum Markov semigroups (QMSs in the sequel) on the von Neumann algebra $\mathcal{A} = \mathcal{B}(\mathsf{h})$, the space of all bounded linear operators on the separable Hilbert spaces $\mathsf{h}$. Particularly, the Hilbert spaces will be of finite dimensional in the examples. It is well known that the generators of norm continuous QMSs have GKSL representations [7, 16, 20]. Still further, as introduced in [9, Theorem 1], we will consider the *special* GKSL representation [28] which is coded in the following theorem.

**Theorem 2.1** *Let $\mathcal{L}$ be the generator of a norm continuous QMS on $\mathcal{B}(\mathsf{h})$ and let $\rho$ be a normal state on $\mathcal{B}(\mathsf{h})$. There exists a bounded self-adjoint operator $H$ and a finite or infinite sequence $(L_l)_{l\geq 1}$ of elements of $\mathcal{B}(\mathsf{h})$ such that*

(i) $\mathrm{tr}(\rho L_l) = 0$ *for all* $l \geq 1$,
(ii) $\sum_{l\geq 1} L_l^* L_l$ *is a strongly convergent sum,*
(iii) *if* $(c_l)_{l\geq 0}$ *is a square summable sequence of complex scalars and* $c_0 \mathbb{1} + \sum_{l\geq 1} c_l L_l = 0$ *then* $c_l = 0$ *for all* $l \geq 0$,
(iv) *the following representation of* $\mathcal{L}$ *holds*

$$\mathcal{L}(x) = i[H, x] - \frac{1}{2}\sum_{l\geq 1}\left(L_l^* L_l x - 2L_l^* x L_l + x L_l^* L_l\right).$$

*If $H'$, $(L_l')_{l\geq 1}$ is another family of bounded operators in $\mathcal{B}(\mathsf{h})$ with $H'$ self-adjoint and the sequence $(L_l')_{l\geq 1}$ is finite or infinite then the conditions (i)–(iv) are fulfilled with $H$, $(L_l)_{l\geq 1}$ replaced by $H'$, $(L_l')_{l\geq 1}$, respectively, if and only if the lengths of the sequences $(L_l)_{l\geq 1}$, $(L_l')_{l\geq 1}$ are equal and for some scalar $c \in \mathbb{R}$ and a unitary matrix $(u_{lj})_{l,j}$ we have*

$$H' = H + c, \quad L_l' = \sum_j u_{lj} L_j. \tag{2.1}$$

Let $\mathcal{T} = (\mathcal{T}_t)_{t\geq 0}$ be a QMS on $\mathcal{A} = \mathcal{B}(\mathsf{h})$ with a faithful normal invariant state $\omega$ with a density matrix $\rho$. Let $(\sigma_t)_{t\in\mathbb{R}}$ be the corresponding modular automorphism: $\sigma_t(x) = \rho^{it} x \rho^{-it}$, $x \in \mathcal{A}$. The dual semigroup of $\mathcal{T}$ is a semigroup $\mathcal{T}'$ that satisfies the relation

$$\omega(\sigma_{i/2}(a)\mathcal{T}_t(b)) = \omega(\sigma_{i/2}(\mathcal{T}_t'(a))b), \quad a, b \in \mathcal{A}, \ t \geq 0, \tag{2.2}$$

or equivalently,

$$\mathrm{tr}\left(\rho^{1/2} a \rho^{1/2} \mathcal{T}_t(b)\right) = \mathrm{tr}\left(\rho^{1/2} \mathcal{T}_t'(a) \rho^{1/2} b\right). \tag{2.3}$$

One can show that $\mathcal{T}'$ is given by [13, Theorem 1]

$$\rho^{1/2} \mathcal{T}_t'(x) \rho^{1/2} = \mathcal{T}_{*t}\left(\rho^{1/2} x \rho^{1/2}\right), \tag{2.4}$$

where $(\mathcal{T}_{*t})_{t\geq 0}$ is the predual semigroup on $\mathcal{A}_*$, the space of trace class operators on $\mathsf{h}$. The semigroup $\mathcal{T}$ is called KMS-symmetric if its dual $\mathcal{T}'$ is equal to itself [13, 15].

In order to discuss the quantum detailed balance (QDB) conditions, we first introduce a reversing operation $\Theta$ [9]; namely, $\Theta : \mathcal{A} \to \mathcal{A}$ is an anti-homomorphic $*$-map ($\Theta(ab) = \Theta(b)\Theta(a)$ and $\Theta(a^*) = \Theta(a)^*$ for all $a, b \in \mathcal{A}$), and also satisfies $\Theta^2 = I$, the identity map on $\mathcal{A}$. In this paper the map $\Theta$ is concretely given by

$$\Theta(a) = \theta a^* \theta, \quad a \in \mathcal{A}, \tag{2.5}$$

where $\theta : \mathsf{h} \to \mathsf{h}$ is the conjugation with respect to a fixed orthonormal basis $(e_n)_{n\geq 0}$ of $\mathsf{h}$ defined as [9]

$$\theta\left(\sum_{n\geq 0} u_n e_n\right) = \sum_{n\geq 0} \overline{u}_n e_n. \tag{2.6}$$

It can be easily shown that the state $\omega$ with a density matrix $\rho$ is invariant under the map $\Theta$, i.e., $\omega(x) = \omega(\Theta(x))$, if and only if $\theta$ commutes with $\rho$, which will be assumed through the paper. In fact, it is the case, if we take the orthonormal basis $(e_n)_{n\geq 0}$ in (2.6) the eigenvectors of $\rho$

Let us introduce the QDB conditions given in [9, 13].

**Definition 2.2** Let $\mathcal{T}$ be a QMS on $\mathcal{A}$ with a dual QMS $\mathcal{T}'$ satisfying the relation (2.2), whose generators are denoted by $\mathcal{L}$ and $\mathcal{L}'$, respectively. The semigroup $\mathcal{T}$ satisfies:

1. the standard quantum detailed balance condition with respect to the reversing operation $\Theta$ (SQDB-$\Theta$) if $\mathcal{T}_t' = \Theta \circ \mathcal{T}_t \circ \Theta$ for all $t \geq 0$,
2. the standard quantum detailed balance condition (SQDB) if the difference of generators $\mathcal{L} - \mathcal{L}'$ is a densely defined derivation.

Here follows some remarks.

***Remark 2.3***

(i) By definition, if $\mathcal{T}$ is KMS-symmetric, i.e., $\mathcal{T} = \mathcal{T}'$, the QMS satisfies SQDB.
(ii) The SQDB-$\Theta$ condition is equivalent to saying that

$$\mathrm{tr}\big(\rho^{1/2} x \rho^{1/2} \mathcal{T}_t(y)\big) = \mathrm{tr}\big(\rho^{1/2} \mathcal{T}_t(\Theta(x)) \rho^{1/2} \Theta(y)\big), \quad x, y \in \mathcal{A}, \ t \geq 0. \tag{2.7}$$

In fact, recall that $\Theta$ is anti-homomorphic and idempotent, and satisfies $\mathrm{tr}(\Theta(x)) = \mathrm{tr}(x)$. The commutativity of $\rho$ and $\theta$ gives the relation $\Theta(x)\rho^{1/2} = \Theta(\rho^{1/2} x)$. Then, the r.h.s. of (2.7) can be rewritten as

$$\begin{aligned}
\mathrm{tr}\big(\rho^{1/2} \mathcal{T}_t(\Theta(x)) \rho^{1/2} \Theta(y)\big) &= \mathrm{tr}\big(\Theta(y)\rho^{1/2} \mathcal{T}_t \circ \Theta(x) \rho^{1/2}\big) \\
&= \mathrm{tr}\big(\Theta(y)\rho^{1/2} \Theta\big(\Theta \circ \mathcal{T}_t \circ \Theta(x)\big)\rho^{1/2}\big) \\
&= \mathrm{tr}\big(\Theta(\rho^{1/2} y)\Theta\big(\rho^{1/2} \Theta \circ \mathcal{T}_t \circ \Theta(x)\big)\big) \\
&= \mathrm{tr}\big(\Theta\big(\rho^{1/2} \Theta \circ \mathcal{T}_t \circ \Theta(x)\rho^{1/2} y\big)\big) \\
&= \mathrm{tr}\big(\rho^{1/2} \Theta \circ \mathcal{T}_t \circ \Theta(x)\rho^{1/2} y\big).
\end{aligned}$$

Therefore, the SQDB-$\Theta$ holds if and only if (2.7) holds. The detailed balance condition SQDB-$\Theta$ here, $\mathrm{tr}(\rho^{1/2} x \rho^{1/2} \mathcal{T}_t(y)) = \mathrm{tr}(\rho^{1/2} \Theta \circ \mathcal{T}_t \circ \Theta(x) \rho^{1/2} y)$, contrasts with the Agarwal-Majewski QDB condition [2, 25]: $\mathrm{tr}(\rho x \mathcal{T}_t(y)) = \mathrm{tr}(\rho(\Theta \circ \mathcal{T}_t \circ \Theta)(x) y)$.

In [13], Fagnola and Umanita characterized the SQDB and SQDB-$\Theta$ conditions by the operators $H$ and $(L_l)_l$ in the special representation of the generator $\mathcal{L}$. Here we introduce it for the SQDB-$\Theta$, which will be used in this paper. In the GKSL representation of the generator, let us put

$$G = -\frac{1}{2}\sum_{l\geq 1} L_l^* L_l - iH \tag{2.8}$$

so that

$$\mathcal{L}(x) = G^* x + \sum_{l\geq 1} L_l^* x L_l + xG. \tag{2.9}$$

**Theorem 2.4** [13, Theorem 8] *A QMS $\mathcal{T}$ satisfies the SQDB-$\Theta$ condition if and only if there exists a special GKSL representation of $\mathcal{L}$ with operators $G$, $L_l$, such that:*

(i) $\rho^{1/2}\Theta(G) = G\rho^{1/2}$,
(ii) $\rho^{1/2}\Theta(L_k) = \sum_j u_{kj} L_j \rho^{1/2}$ *for a self-adjoint unitary $(u_{kj})_{kj}$ on k, a Hilbert space whose dimension is the number of indices for $L_l$'s.*

## 3 Entropy production

In this Section we summarize the entropy production formula for the quantum Markov semigroups obtained by Fagnola and Rebolledo in [9]. Given an invariant state $\rho$ for the QMS $\mathcal{T} = (\mathcal{T}_t)_{t\geq 0}$, the two-point forward and backward states are defined as follows.

**Definition 3.1** The forward two-point state is the normal state on $\mathcal{A}\otimes\mathcal{A}$ given by

$$\overrightarrow{\Omega}_t(a\otimes b) = \mathrm{tr}\big(\rho^{1/2}\Theta(a)\rho^{1/2}\mathcal{T}_t(b)\big), \quad a, b \in \mathcal{A};$$

the backward two-point state is the normal state on $\mathcal{A}\otimes\mathcal{A}$ given by

$$\overleftarrow{\Omega}_t(a\otimes b) = \mathrm{tr}\big(\rho^{1/2}\Theta(\mathcal{T}_t(a))\rho^{1/2}b\big), \quad a, b \in \mathcal{A}.$$

The following was obtained in [9, Proposition 1].

**Proposition 3.2** *Let $\rho = \sum_j \rho_j |e_j\rangle\langle e_j|$ be a spectral decomposition of the invariant state $\rho$ for the QMS $\mathcal{T}$. The density of states $\overrightarrow{\Omega}_0 = \overleftarrow{\Omega}_0$ is the rank one projection*

$$D = |r\rangle\langle r|, \quad r = \sum_j \rho_j^{1/2}\theta e_j \otimes e_j.$$

*The densities of the forward and backward states are, respectively*

$$\overrightarrow{D}_t = (I\otimes\mathcal{T}_{*t})(D), \quad \overleftarrow{D}_t = (\mathcal{T}_{*t}\otimes I)(D).$$

Let us denote by $\mathrm{Tr}(\cdot)$ the trace on $\mathsf{h} \otimes \mathsf{h}$. The relative entropy of $\overrightarrow{\Omega}_t$ with respect to $\overleftarrow{\Omega}_t$ is given by

$$S(\overrightarrow{\Omega}_t, \overleftarrow{\Omega}_t) = \mathrm{Tr}(\overrightarrow{D}_t(\log \overrightarrow{D}_t - \log \overleftarrow{D}_t)),$$

if the support of $\overrightarrow{D}_t$ is included in that of $\overleftarrow{D}_t$ and $+\infty$ otherwise.

**Definition 3.3** The entropy production rate of a QMS $\mathcal{T}$ with an invariant state $\rho$ is defined by

$$\mathbf{ep}(\mathcal{T}, \rho) = \limsup_{t \to 0^+} \frac{S(\overrightarrow{\Omega}_t, \overleftarrow{\Omega}_t)}{t}.$$

In order to have a non-trivial entropy production rate, it is obvious that the supports of $\overrightarrow{D}_t$ and $\overleftarrow{D}_t$ are comparable. So, we consider the following assumption (see [9]).

(**FBS**) *Supports of $\overrightarrow{D}_t$ and $\overleftarrow{D}_t$ coincide and are of finite dimensional.*

Suppose that the QMS has a special representation of the generator $\mathcal{L}$ as in Theorem 2.1. Let $\overrightarrow{\Phi}_*$ and $\overleftarrow{\Phi}_*$ be the linear maps on the trace class operators on $\mathsf{h} \otimes \mathsf{h}$

$$\overrightarrow{\Phi}_*(X) = \sum_l (\mathbb{1} \otimes L_l) X \big(\mathbb{1} \otimes L_l^*\big), \quad \overleftarrow{\Phi}_*(X) = \sum_l (L_l \otimes \mathbb{1}) X \big(L_l^* \otimes \mathbb{1}\big).$$

**Theorem 3.4** ([9, Theorem 5]) *Let $\mathcal{T}$ be a norm continuous QMS on $\mathcal{B}(\mathsf{h})$ with a faithful, normal invariant state $\rho$. Under the assumption (**FBS**) the entropy production is given by*

$$\mathbf{ep}(\mathcal{T}, \rho) = \frac{1}{2} \mathrm{Tr}\Big(\big(\overrightarrow{\Phi}_*(D) - \overleftarrow{\Phi}_*(D)\big)\big(\log \overrightarrow{\Phi}_*(D) - \log \overleftarrow{\Phi}_*(D)\big)\Big).$$

As mentioned in the above, it is important that the supports of $\overrightarrow{D}_t$ and $\overleftarrow{D}_t$ are the same. For it, we will use the following theorem.

**Theorem 3.5** ([9, Theorem 8]) *Let $\mathcal{T}$ be a QMS with generator $\mathcal{L}$ and suppose that $\rho^{1/2}\Theta(G) = G\rho^{1/2}$. The following conditions are equivalent:*

(a) *the closed linear spans of $\{L_l \rho^{1/2} | l \geq 1\}$ and $\{\rho^{1/2}\Theta(L_l) | l \geq 1\}$ in the Hilbert space of Hilbert–Schmidt operators on $\mathsf{h}$ coincide,*
(b) *the forward and backward states $\overrightarrow{D}_t$ and $\overleftarrow{D}_t$ have the same supports.*

## 4 Entropy production of QMS associated with OQWs on the integer lattices with periodic boundary condition

In this Section we consider the QMS associated with OQWs on the integer lattices with periodic boundary condition and compute the entropy production. It will be turned out that the entropy production becomes the weighted sum of the entropy productions in each direction of the axis. To see this we separately consider the model on the cycle and on the torus.

### 4.1 OQWs on the cycle

$B_1$ and $B_{-1}$ are $2 \times 2$ matrices such that

$$\sum_{\alpha \in \{1,-1\}} B_\alpha^* B_\alpha = I_2. \tag{4.1}$$

Let $C_n := \{0, 1, \ldots, n-1\}$ be the cycle of length $n$. Let $\mathsf{h} := l^2(C_n, \mathbb{C}^2) \cong \oplus_{i \in C_n} \mathbb{C}^2$ be the Hilbert space with an orthonormal basis consisting of $e_{(i,\alpha)}, i \in C_n, \alpha = 1, -1$, defined by

$$e_{(i,\alpha)} = (0, \ldots, 0, \overset{i\text{th}}{|\alpha\rangle}, 0, \ldots, 0),$$

where $\{|\alpha\rangle : \alpha = 1, -1\}$ is a canonical basis of $\mathbb{C}^2$, namely $|\alpha\rangle = \begin{cases} (1,0)^{\mathrm{T}}, & \text{if } \alpha = 1, \\ (0,1)^{\mathrm{T}}, & \text{if } \alpha = -1 \end{cases}$.
It is also convenient to consider the isomorphism between $l^2(C_n) \otimes \mathbb{C}^2$ and $\oplus_{i \in C_n} \mathbb{C}^2$ given by the map $u \otimes \xi \mapsto (u_0 \xi, \ldots, u_{n-1}\xi)$, where $u = (u_0, \ldots, u_{n-1}) \in l^2(C_n)$ and $\xi \in \mathbb{C}^2$. In that case we denote by $\{|i\rangle : i \in C_n\}$ the canonical basis of $l^2(C_n)$. Therefore, $e_{(i,\alpha)}$ is identified with $|i\rangle \otimes |\alpha\rangle$.

Let $L_{(i,\alpha)} \in \mathcal{B}(\mathsf{h})$ be defined by

$$L_{(i,\alpha)} = |i + \alpha\rangle\langle i| \otimes B_\alpha.$$

Notice that

$$\sum_{(i,\alpha)} L_{(i,\alpha)}^* L_{(i,\alpha)} = I_{\mathsf{h}}.$$

Let $\mathcal{A} := \oplus_{i \in C_n} \mathcal{B}(\mathbb{C}^2)$ be the direct sum of $2 \times 2$ matrices, which is a von Neumann subalgebra of $\mathcal{B}(\mathsf{h})$. We consider a QMS on $\mathcal{A}$ with a generator given by

$$\begin{aligned} \mathcal{L}(x) &:= -\frac{1}{2} \sum_{(i,\alpha)} \left( L_{(i,\alpha)}^* L_{(i,\alpha)} x - 2 L_{(i,\alpha)}^* x L_{(i,\alpha)} + x L_{(i,\alpha)}^* L_{(i,\alpha)} \right), \\ &= \sum_{(i,\alpha)} L_{(i,\alpha)}^* x L_{(i,\alpha)} - x. \end{aligned} \tag{4.2}$$

Notice that the $G$-operator in (2.8) becomes $G = -\frac{1}{2} \sum_{(i,\alpha)} L_{(i,\alpha)}^* L_{(i,\alpha)} = -\frac{1}{2} I_{\mathsf{h}}$. The action of $\mathcal{L}$ is given as follows. For any $x = (x_i)_{i \in C_n} = \sum_{i \in C_n} |i\rangle\langle i| \otimes x_i \in \mathcal{A}$, we have

$$\mathcal{L}(x) = \left(x_i'\right)_{i\in C_n},$$

with

$$x_i' = B_1^* x_{i+1} B_1 + B_{-1}^* x_{i-1} B_{-1} - x_i, \quad i \in C_n.$$

Let $\theta$ be the conjugation operator on $\mathsf{h}$ defined by

$$\theta\left(\sum_{(i,\alpha)} u_{(i,\alpha)} e_{(i,\alpha)}\right) = \sum_{(i,\alpha)} \overline{u}_{(i,\alpha)} e_{(i,\alpha)}.$$

The reversing operation is then defined on $\mathcal{B}(\mathsf{h})$ by

$$\Theta(a) := \theta a^* \theta, \quad a \in \mathcal{B}(\mathsf{h}). \tag{4.3}$$

**Lemma 4.1** *The invariant states for the QMS $\mathcal{T}$ with generator $\mathcal{L}$ in (4.2) are of the form*

$$\rho = \sum_{i\in C_n} |i\rangle\langle i| \otimes \rho_i$$

*satisfying*

$$\rho_i = B_1 \rho_{i-1} B_1^* + B_{-1} \rho_{i+1} B_{-1}^*, \quad i \in C_n,$$

*where $\rho_i$'s are positive definite $2\times 2$ matrices such that $\sum_{i\in C_n} \mathrm{tr}(\rho_i) = 1$.*

From now on, let us focus on the computation of the entropy production in this model. For a simplicity we will require that the normalized identity operator on $\mathsf{h}$ is an invariant state (may not be unique) for the QMS. By Lemma 4.1, a sufficient condition, which will be assumed, is the relation

$$B_1 B_1^* + B_{-1} B_{-1}^* = I_2. \tag{4.4}$$

In order to have nontrivial entropy production, it is also required to satisfy the condition (a) in Theorem 3.5. Therefore, we define the operators $B_{\pm 1}$ as follows: given any $2\times 2$ matrix $U$ and a complex number $\lambda \neq 0$, let

$$B_1 = U, \quad B_{-1} = \lambda U^{\mathrm{T}}. \tag{4.5}$$

To satisfy the conditions (4.1) and (4.4), $U$ and $\lambda$ should fulfill

$$U^*U + |\lambda|^2 (UU^*)^{\mathrm{T}} = I_2, \quad UU^* + |\lambda|^2 (U^*U)^{\mathrm{T}} = I_2. \tag{4.6}$$

In the sequel we assume (4.6). Then, the state $\rho = \frac{1}{2n} I_{\mathsf{h}}$ is an invariant state for the QMS generated by $\mathcal{L}$ in (4.2). Let us provide with some necessary lemmas.

**Lemma 4.2** *Let $r$ be the vector of $\mathsf{h} \otimes \mathsf{h}$ given in Proposition 3.2. Then we have*

$$\langle (\mathbb{1} \otimes L_{(i,\alpha)})r, (\mathbb{1} \otimes L_{(j,\beta)})r \rangle = \delta_{i,j}\delta_{\alpha,\beta}\frac{1}{2n}\operatorname{tr}\big(B_\alpha^* B_\beta\big),$$
$$\langle (L_{(i,\alpha)} \otimes \mathbb{1})r, (L_{(j,\beta)} \otimes \mathbb{1})r \rangle = \delta_{i,j}\delta_{\alpha,\beta}\frac{1}{2n}\operatorname{tr}\big(B_\alpha^* B_\beta\big).$$

***Proof*** Since $r = \frac{1}{\sqrt{2n}} \sum_{(k,\gamma)} e_{(k,\gamma)} \otimes e_{(k,\gamma)}$

$$\begin{aligned}
(\mathbb{1} \otimes L_{(i,\alpha)})r &= \frac{1}{\sqrt{2n}} \sum_{(k,\gamma)} e_{(k,\gamma)} \otimes L_{(i,\alpha)} e_{(k,\gamma)} \\
&= \frac{1}{\sqrt{2n}} \sum_{(k,\gamma)} e_{(k,\gamma)} \otimes (|i+\alpha\rangle\langle i| \otimes B_\alpha)\,(|k\rangle \otimes |\gamma\rangle) \\
&= \frac{1}{\sqrt{2n}} \sum_{\gamma=\pm 1} e_{(i,\gamma)} \otimes (|i+\alpha\rangle \otimes B_\alpha|\gamma\rangle)
\end{aligned}$$

Therefore,

$$\begin{aligned}
\langle (\mathbb{1} \otimes L_{(i,\alpha)})r, (\mathbb{1} \otimes L_{(j,\beta)})r \rangle &= \frac{1}{2n} \sum_{\gamma=\pm 1} \delta_{i,j}\delta_{\alpha,\beta}\langle \gamma, \big(B_\alpha^* B_\beta\big)\gamma \rangle \\
&= \delta_{i,j}\delta_{\alpha,\beta}\frac{1}{2n}\operatorname{tr}\big(B_\alpha^* B_\beta\big).
\end{aligned}$$

The second equality can be shown similarly. □

For a notational simplicity, let us denote

$$u_{(i,\alpha)} = \frac{(\mathbb{1} \otimes L_{(i,\alpha)})r}{\|(\mathbb{1} \otimes L_{(i,\alpha)})r\|} \text{ and } v_{(i,\alpha)} = \frac{(L_{(i,\alpha)} \otimes \mathbb{1})r}{\|(L_{(i,\alpha)} \otimes \mathbb{1})r\|}, \quad i \in C_n,\ \alpha = \pm 1.$$

**Lemma 4.3** *For all $i, j \in C_n$ and $\alpha, \beta \in \{+1, -1\}$,*

$$\begin{aligned}
&\operatorname{Tr}(|u_{(i,\alpha)}\rangle\langle u_{(i,\alpha)}|\,|v_{(j,\beta)}\rangle\langle v_{(j,\beta)}|) \\
&\quad = \delta_{i,j+\beta}\delta_{i+\alpha,j}\frac{1}{\operatorname{tr}\big(B_\alpha^* B_\alpha\big)\operatorname{tr}\big(B_\beta^* B_\beta\big)}\Big|\sum_{\gamma,\gamma' \in \{+1,-1\}} \langle \gamma, B_\alpha^* \gamma' \rangle \langle \gamma, B_\beta \gamma' \rangle\Big|^2
\end{aligned}$$

***Proof*** Since $|u_{(i,\alpha)}\rangle\langle u_{(i,\alpha)}|$ and $|v_{(j,\beta)}\rangle\langle v_{(j,\beta)}|$ are, respectively, rank one projections,

$$\operatorname{Tr}(|u_{(i,\alpha)}\rangle\langle u_{(i,\alpha)}|\,|v_{(j,\beta)}\rangle\langle v_{(j,\beta)}|) = |\langle u_{(i,\alpha)}, v_{(j,\beta)}\rangle|^2.$$

As was shown in the proof of Lemma 4.2, we have

$$(\mathbb{1}\otimes L_{(i,\alpha)})r = \frac{1}{\sqrt{2n}}\sum_{\gamma=\pm 1} e_{(i,\gamma)}\otimes(|i+\alpha\rangle\otimes B_\alpha|\gamma\rangle)$$

$$(L_{(j,\beta)}\otimes\mathbb{1})r = \frac{1}{\sqrt{2n}}\sum_{\gamma'=\pm 1}(|j+\beta\rangle\otimes B_\beta|\gamma'\rangle)\otimes e_{(j,\gamma')}.$$

Taking the inner product and implementing the norms of $(\mathbb{1}\otimes L_{(i,\alpha)})r$ and $(L_{(j,\beta)}\otimes\mathbb{1})r$ from Lemma 4.2, we get the result. □

We are ready for the computation of the entropy production for the QMS associated with open quantum walks on a cycle. We have

**Theorem 4.4** *Suppose that $B_{\pm 1}$ are defined by (4.5) and the relation (4.6) is satisfied. Then, the entropy production for the QMS associated with OQW on the cycle with respect to the invariant state $\rho = \frac{1}{2n}I_h$ is given by*

$$\mathbf{ep}(\mathcal{T},\rho) = p\log\frac{p}{q} + q\log\frac{q}{p},\ \textit{with } p = \frac{1}{1+|\lambda|^2}\ \textit{and } q = \frac{|\lambda|^2}{1+|\lambda|^2}. \tag{4.7}$$

***Proof*** By Lemma 4.2, using $D = |r\rangle\langle r|$,

$$\begin{aligned}\overrightarrow{\Phi}_*(D) &= \sum_{(i,\alpha)}\|(\mathbb{1}\otimes L_{(i,\alpha)})r\|^2|u_{(i,\alpha)}\rangle\langle u_{(i,\alpha)}|\\ &= \sum_{(i,\alpha)}\frac{1}{2n}\mathrm{tr}\big(B_\alpha^*B_\alpha\big)|u_{(i,\alpha)}\rangle\langle u_{(i,\alpha)}|.\end{aligned} \tag{4.8}$$

Similarly we have

$$\overleftarrow{\Phi}_*(D) = \sum_{(i,\alpha)}\frac{1}{2n}\mathrm{tr}\big(B_\alpha^*B_\alpha\big)|v_{(i,\alpha)}\rangle\langle v_{(i,\alpha)}|. \tag{4.9}$$

By Lemma 4.2, $\{|u_{(i,\alpha)}\rangle\langle u_{(i,\alpha)}| : i\in C_n,\ \alpha=\pm1\}$ and $\{|v_{(i,\alpha)}\rangle\langle v_{(i,\alpha)}| : i\in C_n,\ \alpha=\pm1\}$ are mutually orthogonal projections, respectively. Therefore, by (4.8) and (4.9),

$$\begin{aligned}\mathbf{ep}(\mathcal{T},\rho) &= \tfrac{1}{2}\mathrm{Tr}((\overrightarrow{\Phi}_*(D)-\overleftarrow{\Phi}_*(D))(\log(\overrightarrow{\Phi}_*(D))-\log(\overleftarrow{\Phi}_*(D))))\\ &= \tfrac{1}{2}\mathrm{Tr}\Bigg(\Bigg(\textstyle\sum_{(i,\alpha)}\frac{\mathrm{tr}\left(B_\alpha^*B_\alpha\right)}{2n}|u_{(i,\alpha)}\rangle\langle u_{(i,\alpha)}| - \sum_{(j,\beta)}\frac{\mathrm{tr}\left(B_\beta^*B_\beta\right)}{2n}|v_{(j,\beta)}\rangle\langle v_{(j,\beta)}|\Bigg)\\ &\quad\Bigg(\textstyle\sum_{(i,\alpha)}\log\left(\frac{\mathrm{tr}\left(B_\alpha^*B_\alpha\right)}{2n}\right)|u_{(i,\alpha)}\rangle\langle u_{(i,\alpha)}| - \sum_{(j,\beta)}\log\left(\frac{\mathrm{tr}\left(B_\beta^*B_\beta\right)}{2n}\right)|v_{(j,\beta)}\rangle\langle v_{(j,\beta)}|\Bigg)\Bigg)\\ &= \textstyle\sum_{(i,\alpha)}\frac{\mathrm{tr}\left(B_\alpha^*B_\alpha\right)}{2n}\log\frac{\mathrm{tr}\left(B_\alpha^*B_\alpha\right)}{2n} - \frac{1}{2}\sum_{(i,\alpha)}\sum_{(j,\beta)}\Bigg(\frac{\mathrm{tr}\left(B_\alpha^*B_\alpha\right)}{2n}\log\frac{\mathrm{tr}\left(B_\beta^*B_\beta\right)}{2n}\\ &\quad+\textstyle\frac{\mathrm{tr}\left(B_\beta^*B_\beta\right)}{2n}\log\frac{\mathrm{tr}\left(B_\alpha^*B_\alpha\right)}{2n}\Bigg)\mathrm{Tr}(|u_{(i,\alpha)}\rangle\langle u_{(i,\alpha)}|v_{(j,\beta)}\rangle\langle v_{(j,\beta)}|).\end{aligned}$$

In the last term of trace, by Lemma 4.3, given $(i, \alpha)$ a non-zero term appears only for the index $(j, \beta) = (i + \alpha, -\alpha)$. Implementing the value of the trace by using Lemma 4.3 we get

$$\begin{aligned}\mathbf{ep}(\mathcal{T}, \rho) &= \tfrac{1}{2} \textstyle\sum_{\alpha \in \{+1,-1\}} \mathrm{tr}\big(B_\alpha^* B_\alpha\big) \log \frac{\mathrm{tr}\left(B_\alpha^* B_\alpha\right)}{2n} \\ &\quad - \tfrac{1}{4} \textstyle\sum_{\alpha \in \{+1,-1\}} \left( \frac{1}{\mathrm{tr}\left(B_{-\alpha}^* B_{-\alpha}\right)} \log \frac{\mathrm{tr}\left(B_{-\alpha}^* B_{-\alpha}\right)}{2n} + \frac{1}{\mathrm{tr}\left(B_\alpha^* B_\alpha\right)} \log \frac{\mathrm{tr}\left(B_\alpha^* B_\alpha\right)}{2n} \right) \\ &\quad \times \Big| \textstyle\sum_{\gamma, \gamma' \in \{+1,-1\}} \langle \gamma, B_{-\alpha} \gamma' \rangle \langle \gamma, B_\alpha^* \gamma' \rangle \Big|^2 .\end{aligned}$$

By the definition $B_1 = U,\ B_{-1} = \lambda U^{\mathrm{T}}$, the last line becomes

$$\left| \sum_{\gamma, \gamma' \in \{+1,-1\}} \langle \gamma, B_{-\alpha} \gamma' \rangle \langle \gamma, B_\alpha^* \gamma' \rangle \right|^2 = |\lambda|^2 \mathrm{tr}\big(U^* U\big)^2,$$

no matter what the value $\alpha \in \{1, -1\}$ is. Putting this into the formula we get

$$\mathbf{ep}(\mathcal{T}, \rho) = \frac{1}{2} \mathrm{tr}\big(U^* U\big) \Big( \log \frac{1}{|\lambda|^2} + |\lambda|^2 \log |\lambda|^2 \Big). \tag{4.10}$$

Now taking traces to both sides of (4.6), we obtain $\mathrm{tr}(U^* U) = 2/(1 + |\lambda|^2)$. Putting back into (4.10) and defining $p = 1/(1 + |\lambda|^2)$ and $q = |\lambda|^2/(1 + |\lambda|^2)$, we get

$$\begin{aligned}\mathbf{ep}(\mathcal{T}, \rho) &= \frac{1}{1 + |\lambda|^2} \Big( \log \frac{1}{|\lambda|^2} + |\lambda|^2 \log |\lambda|^2 \Big) \\ &= p \log \frac{p}{q} + q \log \frac{q}{p}.\end{aligned}$$

It completes the proof. □

Let us consider some examples.

***Example 4.5*** Let $B_1 = \sqrt{p} U$ and $B_{-1} = \sqrt{q} U^{\mathrm{T}}$ (or $B_- = -\sqrt{q} U^{\mathrm{T}}$) with $p, q \in (0, 1)$, $p + q = 1$, where $U$ is a real unitary matrix. In this case we have

$$U = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \text{ or } U = \begin{pmatrix} a & b \\ b & -a \end{pmatrix} \text{ with } a, b \in \mathbb{R},\ a^2 + b^2 = 1. \tag{4.11}$$

Directly computing from Theorem 4.4, we get

$$\mathbf{ep}(\mathcal{T}, \rho) = p \log \frac{p}{q} + q \log \frac{q}{p}.$$

Notice that if we define two probability measures $P$ and $Q$ on $\{+1, -1\}$ by

$$P(+1) = p,\ P(-1) = q,\ \text{and } Q(+1) = q,\ Q(-1) = p,$$

the entropy production is the classical relative entropy $H(Q|P)$:

$$H(Q|P) = \sum_{\alpha\in\{+1,-1\}} Q(\alpha)\log\frac{Q(\alpha)}{P(\alpha)}.$$

***Example 4.6*** $B_1 = \begin{pmatrix} a & b \\ 0 & -a \end{pmatrix}$, $B_{-1} = (B_1)^{\mathrm{T}}$, with $a, b \in \mathbb{R}$ satisfying $2a^2+b^2 = 1$. In this case we promptly get $\mathbf{ep}(\mathcal{T}, \rho) = 0$, since $p = q = 1/2$ in (4.7).

### 4.2 QMS of open quantum walks on the torus

In this subsection we consider the higher dimensional model. We focus on a two dimensional torus and will see that it can be easily extended to more higher dimensions. Let

$$\mathbb{T} \equiv \mathbb{T}_{(m,n)} := \mathbb{Z}^2/(m\mathbb{Z}\times n\mathbb{Z}) = \{0, 1, \ldots, m-1\}\times\{0, 1, \ldots, n-1\}$$

be a torus of size $m\times n$. The Hilbert space for our model is $\mathsf{h} = \oplus_{\mathbf{i}\in\mathbb{T}}\mathbb{C}^4 \cong l^2(\mathbb{T})\otimes\mathbb{C}^4$. The canonical basis of $l^2(\mathbb{T})$ is denoted by $\{|\mathbf{i}\rangle : \mathbf{i}\in\mathbb{T}\}$ and we use $\{|e_\alpha\rangle : \alpha = \pm1, \pm2\}$ for the canonical basis of $\mathbb{C}^4$. Hence the set $\{e_{(\mathbf{i},\alpha)} := |\mathbf{i}\rangle\otimes|e_\alpha\rangle : \mathbf{i}\in\mathbb{T}, \alpha = \pm1, \pm2\}$ constitutes an orthonormal basis for $\mathsf{h}$.

Let $U$ and $V$ be (complex) $4\times4$ matrices and $\mu$ and $\nu$ be non-zero complex numbers. For $\alpha = \pm1, \pm2$, define the following $4\times4$ matrices $B_\alpha$:

$$B_1 = U,\ \ B_{-1} = \mu U^{\mathrm{T}} \text{ and } B_2 = V,\ \ B_{-2} = \nu V^{\mathrm{T}}. \tag{4.12}$$

We require the following conditions:

$$\sum_{\alpha=\pm1,\pm2} B_\alpha^* B_\alpha = I_4 \text{ and } \sum_{\alpha=\pm1,\pm2} B_\alpha B_\alpha^* = I_4. \tag{4.13}$$

For it, it is necessary and sufficient that $U$, $V$ and $\mu$, $\nu$ satisfy:

$$U^*U + V^*V + |\mu|^2(UU^*)^{\mathrm{T}} + |\nu|^2(VV^*)^{\mathrm{T}} = I_4, \tag{4.14}$$
$$UU^* + VV^* + |\mu|^2(U^*U)^{\mathrm{T}} + |\nu|^2(V^*V)^{\mathrm{T}} = I_4. \tag{4.15}$$

Notice that we can find many pairs $\{U, V\}$ and $\{\mu, \nu\}$ that satisfy the Eqs. (4.14) and (4.15). For example, take $U$ and $V$ as the following form,

$$U = \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix},\ \ V = \begin{pmatrix} 0 & 0 \\ 0 & C \end{pmatrix},$$

where $B$ and $C$ are $2\times2$ matrices, and the pairs $(B, \mu)$ and $(C, \nu)$ take the role of the pair $(U, \lambda)$ in (4.5).

For each element $e_\alpha$, $\alpha = \pm 1, \pm 2$, we assign a unit vector $\widehat{\theta}(e_\alpha)$ in $\mathbb{R}^2$ by

$$\widehat{\theta}(e_1) = (1, 0), \; \widehat{\theta}(e_2) = (0, 1), \; \widehat{\theta}(e_{-1}) = -\widehat{\theta}(e_1), \; \widehat{\theta}(e_{-2}) = -\widehat{\theta}(e_2).$$

For each pair $(\mathbf{i}, \alpha) \in \mathbb{T} \times \{\pm 1, \pm 2\}$, define a linear operator $L_{(\mathbf{i},\alpha)} : \mathsf{h} \to \mathsf{h}$ by

$$L_{(\mathbf{i},\alpha)} := |\mathbf{i} + \widehat{\theta}(e_\alpha)\rangle\langle \mathbf{i}| \otimes B_\alpha. \tag{4.16}$$

We can check

$$\sum_{(\mathbf{i},\alpha)} L^*_{(\mathbf{i},\alpha)} L_{(\mathbf{i},\alpha)} = I_{\mathsf{h}} \text{ and } \sum_{(\mathbf{i},\alpha)} L_{(\mathbf{i},\alpha)} L^*_{(\mathbf{i},\alpha)} = I_{\mathsf{h}}. \tag{4.17}$$

As like the model on the cycle, the generator of the QMS associated with the open quantum walk on the torus is defined as follows: for $x \in \mathcal{A} := \oplus_{\mathbf{i} \in \mathbb{T}} \mathcal{B}(\mathbb{C}^4) \subset \mathcal{B}(\mathsf{h})$,

$$\begin{aligned}
\mathcal{L}(x) &:= -\frac{1}{2} \sum_{(\mathbf{i},\alpha)} \left( L^*_{(\mathbf{i},\alpha)} L_{(\mathbf{i},\alpha)} x - 2 L^*_{(\mathbf{i},\alpha)} x L_{(\mathbf{i},\alpha)} + x L^*_{(\mathbf{i},\alpha)} L_{(\mathbf{i},\alpha)} \right) \\
&= \sum_{(\mathbf{i},\alpha)} L^*_{(\mathbf{i},\alpha)} x L_{(\mathbf{i},\alpha)} - x \\
&=: G^* x + \sum_{(\mathbf{i},\alpha)} L^*_{(\mathbf{i},\alpha)} x L_{(\mathbf{i},\alpha)} + x G,
\end{aligned} \tag{4.18}$$

with $G = -\frac{1}{2} \sum_{(\mathbf{i},\alpha)} L^*_{(\mathbf{i},\alpha)} L_{(\mathbf{i},\alpha)} = -\frac{1}{2} I_{\mathsf{h}}$. Notice that by (4.13), the state $\rho := \frac{1}{4mn} I_{\mathsf{h}}$ is an invariant state for the QMS.

Constructing the QMS $\mathcal{T} = (\mathcal{T}_t)_{t \geq 0}$, let us compute the entropy production. The definition of forward and backward two-point states is the same as given by Definition 3.1. Defining a unit vector

$$r = \frac{1}{\sqrt{4mn}} \sum_{(\mathbf{i},\alpha) \in \mathbb{T} \times \{\pm 1, \pm 2\}} e_{(\mathbf{i},\alpha)} \otimes e_{(\mathbf{i},\alpha)} \in \mathsf{h} \otimes \mathsf{h}$$

and a rank one projection $D := |r\rangle\langle r| \in \mathcal{B}(\mathsf{h}) \otimes \mathcal{B}(\mathsf{h})$, our construction of the QMS is to satisfy the assumption (**FBS**). Therefore, we can use the formula for the entropy production in Theorem 3.4. To state the result let us introduce some convenient notions. By (4.14), taking traces we get

$$(1 + |\mu|^2) \mathrm{tr}\big(U^* U\big) + (1 + |\nu|^2) \mathrm{tr}\big(V^* V\big) = 4.$$

Let $\lambda_1, \lambda_2 \in [0, 1]$ such that $\lambda_1 + \lambda_2 = 1$ be defined as

$$(1 + |\mu|^2) \mathrm{tr}\big(U^* U\big) = 4\lambda_1 \text{ and } (1 + |\nu|^2) \mathrm{tr}\big(V^* V\big) = 4\lambda_2. \tag{4.19}$$

Let us also define two Bernoulli parameters:

$$p_\mu = \frac{1}{1 + |\mu|^2}, \; q_\mu = \frac{|\mu|^2}{1 + |\mu|^2} \text{ and } p_\nu = \frac{1}{1 + |\nu|^2}, \; q_\nu = \frac{|\nu|^2}{1 + |\nu|^2}. \tag{4.20}$$

**Theorem 4.7** *Let $\mathcal{T} = (\mathcal{T}_t)_{t\geq 0}$ be the QMS with a generator given by (4.18). Then, the entropy production for the QMS is given by*

$$\boldsymbol{ep}(\mathcal{T}, \rho) = \lambda_1 \left( q_\mu \log \frac{q_\mu}{p_\mu} + p_\mu \log \frac{p_\mu}{q_\mu} \right) + \lambda_2 \left( q_\nu \log \frac{q_\nu}{p_\nu} + p_\nu \log \frac{p_\nu}{q_\nu} \right).$$

In order to prove the theorem, similarly to the previous subsection, we prepare some useful lemmas.

**Lemma 4.8** *The following orthogonalities hold: for all $\boldsymbol{i}, \boldsymbol{j} \in \mathbb{T}$ and $\alpha, \beta \in \{\pm 1, \pm 2\}$,*

$$\langle (\mathbb{1} \otimes L_{(\boldsymbol{i},\alpha)})r, (\mathbb{1} \otimes L_{(\boldsymbol{j},\beta)})r \rangle = \delta_{\boldsymbol{i},\boldsymbol{j}} \delta_{\alpha,\beta} \frac{1}{4mn} \mathrm{tr}\big(B_\alpha^* B_\beta\big),$$

$$\langle (L_{(\boldsymbol{i},\alpha)} \otimes \mathbb{1})r, (L_{(\boldsymbol{j},\beta)} \otimes \mathbb{1})r \rangle = \delta_{\boldsymbol{i},\boldsymbol{j}} \delta_{\alpha,\beta} \frac{1}{4mn} \mathrm{tr}\big(B_\alpha^* B_\beta\big).$$

***Proof*** It can be shown by the same way done in the proof of Lemma 4.2. □

Let us define the unitary vectors.

$$u_{(\mathbf{i},\alpha)} = \frac{(\mathbb{1} \otimes L_{(\mathbf{i},\alpha)})r}{\|(\mathbb{1} \otimes L_{(\mathbf{i},\alpha)})r\|} \text{ and } v_{(\mathbf{i},\alpha)} = \frac{(L_{(\mathbf{i},\alpha)} \otimes \mathbb{1})r}{\|(L_{(\mathbf{i},\alpha)} \otimes \mathbb{1})r\|}, \quad \mathbf{i} \in \mathbb{T},\ \alpha = \pm 1, \pm 2.$$

Then the sets of vectors $\{u_{(\mathbf{i},\alpha)}\}_{(\mathbf{i},\alpha)}$ and $\{v_{(\mathbf{i},\alpha)}\}_{(\mathbf{i},\alpha)}$ are orthonormal systems in $\mathsf{h} \otimes \mathsf{h}$. Using Lemma 4.8 we have the representations:

**Lemma 4.9**

$$\overrightarrow{\Phi}_*(D) = \sum_{(\boldsymbol{i},\alpha)\in\mathbb{T}\times\{\pm 1,\pm 2\}} \frac{1}{4mn} \mathrm{tr}\big(B_\alpha^* B_\alpha\big) |u_{(\boldsymbol{i},\alpha)}\rangle\langle u_{(\boldsymbol{i},\alpha)}|,$$

$$\overleftarrow{\Phi}_*(D) = \sum_{(\boldsymbol{i},\alpha)\in\mathbb{T}\times\{\pm 1,\pm 2\}} \frac{1}{4mn} \mathrm{tr}\big(B_\alpha^* B_\alpha\big) |v_{(\boldsymbol{i},\alpha)}\rangle\langle v_{(\boldsymbol{i},\alpha)}|,$$

We will need also

**Lemma 4.10** *For all $\boldsymbol{i}, \boldsymbol{j} \in \mathbb{T}$ and $\alpha, \beta \in \{\pm 1, \pm 2\}$,*

$$\langle u_{(\boldsymbol{i},\alpha)}, v_{(\boldsymbol{j},\beta)} \rangle = \delta_{\boldsymbol{i},\boldsymbol{j}+\widehat{\theta}(e_\beta)} \delta_{\boldsymbol{i}+\widehat{\theta}(e_\alpha),\boldsymbol{j}} \frac{1}{\sqrt{\mathrm{tr}\big(B_\alpha^* B_\alpha\big)\mathrm{tr}\big(B_\beta^* B_\beta\big)}} \mathrm{tr}\big(B_\alpha^* B_\beta^{\mathrm{T}}\big).$$

***Proof*** By the same lines for the proof of Lemma 4.3 we get

$$\begin{aligned} \langle u_{(\mathbf{i},\alpha)}, v_{(\mathbf{j},\beta)} \rangle &= \delta_{\mathbf{i},\mathbf{j}+\widehat{\theta}(e_\beta)} \delta_{\mathbf{i}+\widehat{\theta}(e_\alpha),\mathbf{j}} \frac{1}{\sqrt{\mathrm{tr}\big(B_\alpha^* B_\alpha\big)\mathrm{tr}\big(B_\beta^* B_\beta\big)}} \\ &\quad \sum_{\gamma,\gamma'\in\{\pm 1,\pm 2\}} \big\langle e_\gamma, B_\alpha^* e_{\gamma'} \big\rangle \big\langle e_\gamma, B_\beta e_{\gamma'} \big\rangle. \end{aligned}$$

Then observe that

$$\sum_{\gamma,\gamma'\in\{\pm1,\pm2\}} \langle e_\gamma, B_\alpha^* e_{\gamma'}\rangle\langle e_\gamma, B_\beta e_{\gamma'}\rangle = \sum_{\gamma,\gamma'\in\{\pm1,\pm2\}} \langle e_\gamma, B_\alpha^* e_{\gamma'}\rangle\langle e'_\gamma, B_\beta^{\mathrm{T}} e_\gamma\rangle$$
$$= \mathrm{tr}\big(B_\alpha^* B_\beta^{\mathrm{T}}\big).$$

The proof is completed. □

***Proof of Theorem 4.7*** We use Lemmas 4.9 and 4.10 in the formula of entropy production in Theorem 3.4 and get

$$\begin{aligned}
\mathbf{ep}(\mathcal{T},\rho) = {}& \frac{1}{4}\Big(\mathrm{tr}\big(U^*U\big)\log\frac{\mathrm{tr}\big(U^*U\big)}{4mn} + |\mu|^2\mathrm{tr}\big(U^*U\big)\log\frac{|\mu|^2\mathrm{tr}\big(U^*U\big)}{4mn}\\
&+\mathrm{tr}\big(V^*V\big)\log\frac{\mathrm{tr}\big(V^*V\big)}{4mn} + |\nu|^2\mathrm{tr}\big(V^*V\big)\log\frac{|\nu|^2\mathrm{tr}\big(V^*V\big)}{4mn}\\
&-\frac{1}{4}\Big(\mathrm{tr}\big(U^*U\big)\log\frac{|\mu|^2\mathrm{tr}\big(U^*U\big)}{4mn} + |\mu|^2\mathrm{tr}\big(U^*U\big)\log\frac{\mathrm{tr}\big(U^*U\big)}{4mn}\\
&+\mathrm{tr}\big(V^*V\big)\log\frac{|\nu|^2\mathrm{tr}\big(V^*V\big)}{4mn} + |\nu|^2\mathrm{tr}\big(V^*V\big)\log\frac{\mathrm{tr}\big(V^*V\big)}{4mn}\Big)\\
={}& \frac{1}{4}\Big(\mathrm{tr}\big(U^*U\big)\Big(\log\frac{1}{|\mu|^2} + |\mu|^2\log|\mu|^2\Big)\\
&+\mathrm{tr}\big(V^*V\big)\Big(\log\frac{1}{|\nu|^2} + |\nu|^2\log|\nu|^2\Big)\Big).
\end{aligned}$$

Using the notations in (4.19) and (4.20), the last term can be rewritten as the formula in the statement. □

## 5 QMS of open quantum walks on the crystal torus

In this section we consider the model of QMSs coming from open quantum walks on the crystal lattices. The crystal lattices can be thought of as the regular lattices but each lattice point may have some intrinsic structure. The integer lattices are definitely special class of crystal lattices and in general when we consider some model on the crystal lattices it gives rich properties more than that could be obtained on the integer lattices. The central limit theorem for open quantum walks on the crystal lattices was obtained in [17] and here, for self-containedness, we briefly introduce how the crystal lattices can be constructed and define the QMS associated with the OQWs on a finite system, say on a crystal torus. See the reference [17] for more details.

### 5.1 Construction of crystal lattices

Let $G_0 = (V_0, E_0)$ be a finite graph which may have multi-edges and self-loops. We use the notation $A(G_0)$ for the set of symmetric arcs induced by $E_0$. (An arc is an edge with a direction. So, each edge of $E_0$ gives a pair of arcs $e$ and $\overline{e}$, where $\overline{e}$ is an arc having the same end points of $e$ but with a reversed direction.) The homology group of $G_0$ with integer coefficients is denoted by $H_1(G_0, \mathbb{Z})$. Let the set of basis of $H_1(G_0, \mathbb{Z})$ be $\{C_1, C_2, \ldots, C_{b_1}\}$ corresponding to the fundamental cycles of $G_0$, where $b_1$ is the first Betti number of $G_0$. The spanning tree induced by $\{C_1, C_2, \ldots, C_{b_1}\}$ is denoted by $\mathbb{T}_0$. Then $A(\mathbb{T}_0)^c$ can be indexed as $\{e_i, \overline{e}_i : i = 1, \ldots, b_1\}$ so that each $C_i$ is the cycle generated by adding $e_i$ to $\mathbb{T}_0$. Given a subgroup $H \subset H_1(G_0, \mathbb{Z})$, the quotient group $H_1(G_0, \mathbb{Z})/H$ induces an abstract periodic lattice $\mathbb{L}$, which is isomorphic to an integer lattice $\mathbb{Z}^d$ for some $1 \le d \le b_1$ [29]. Let $\{\widehat{\theta}_i : i = 1, \ldots, d\} \subset \mathbb{R}^d$ be a basis that generates $\mathbb{L}$, i.e.,

$$\mathbb{L} = \left\{ \sum n_i \widehat{\theta}_i : n_i \in \mathbb{Z},\ i = 1, \ldots, d \right\}.$$

We choose a map $\widehat{\theta}$ from $A(G_0)$ to $\{\pm\widehat{\theta}_i : i = 1, \ldots, d\} \cup \{0\}$ that satisfies (i) $\widehat{\theta}(\overline{e}) = -\widehat{\theta}(e)$, (ii) the rank of $\{\widehat{\theta}(e) : e \in A(G_0)\}$ is $d$, and (iii) $\widehat{\theta}(e) = 0$ if and only if $e \in A(\mathbb{T}_0)$.

The crystal lattice of a fundamental finite graph $G(V_0, E_0)$ is the covering graph $G(V, A)$ defined as follows. Let $\phi_0(\mathbb{T}_0)$ be a realization of $\mathbb{T}_0$ on $\mathbb{R}^d$, namely, $\phi_0(\mathbb{T}_0)$ is a finite graph on $\mathbb{R}^d$ with vertices $\phi_0(u)$, $u \in \mathbb{T}_0$ and the edges of $\phi_0(\mathbb{T}_0)$ are the line segments connecting the points of $\phi_0(V_0)$ such that $\phi_0(u) \sim \phi_0(v)$ if and only if $u \sim v$. Then the vertex set $V$ of $G(V, A)$ is defined by

$$V = \mathbb{L} + \phi_0(V_0).$$

We use a simple notation $(\mathbf{i}, u)$ for the element $\mathbf{i} + \phi_0(u)$ for some $\mathbf{i} \in \mathbb{L}$ and $u \in V_0$. Next the edges are defined by

$$A = \cup_{\mathbf{i}\in\mathbb{L}} \left\{ \left( (\mathbf{i}, \mathrm{o}(e)), (\mathbf{i} + \widehat{\theta}(e), \mathrm{t}(e)) \right) \mid e \in A(G_0) \right\}.$$

In Fig. 1 we see a hexagonal lattice. In the fundamental finite graph $G_0$ we have two generating cycles, e.g. $C_1 = e_1 + \overline{e}_3$ and $C_2 = e_2 + \overline{e}_3$. For a subgroup $H = \{0\}$ we have $d = 2$. We have chosen $\widehat{\theta}_1 = (1, 0)$ and $\widehat{\theta}_2 = (0, 1)$. $V(\mathbb{T}_0) = \{u, v\}$ and $A(\mathbb{T}_0) = \{e_3, \overline{e}_3\}$. We have taken $\widehat{\theta}(e_i) = \widehat{\theta}_i$, $i = 1, 2$.

### 5.2 QMS associated with open quantum walks on the crystal torus

In this subsection we consider open quantum walks on a crystal torus, i.e., 2-dimensional finite crystals with periodic boundary conditions. Instead of dealing with the general models, to make everything concrete, we just consider a hexagonal torus. For it let $m, n$ be given natural numbers and let

$$\mathbb{T} \equiv \mathbb{T}_{(m,n)} := \mathbb{Z}^2/(m\mathbb{Z} \times n\mathbb{Z}) = \{0, 1, \ldots, m-1\} \times \{0, 1, \ldots, n-1\}$$

be a torus of size $m \times n$. The Hilbert space for our model is $\mathsf{h} = \oplus_{\mathbf{i}\in\mathbb{T}} \mathbb{C}^6 \cong l^2(\mathbb{T}) \otimes \mathbb{C}^6$.

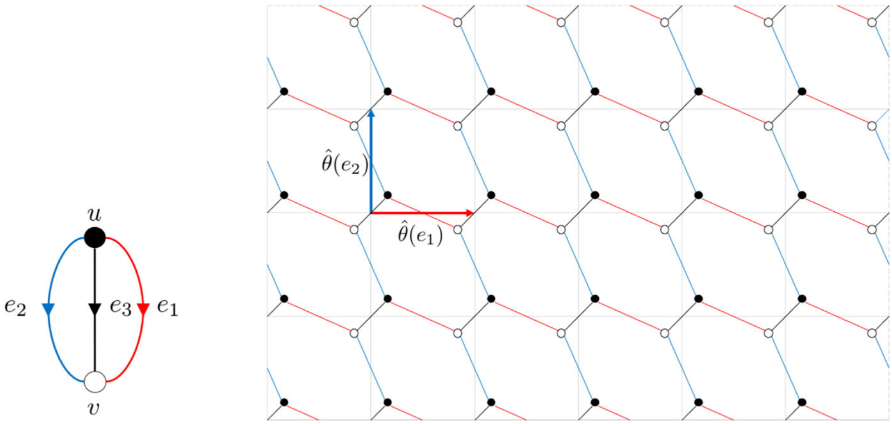


**Fig. 1** Hexagonal lattice: underlying graph $G_0$ for hexagonal lattice (left), hexagonal lattice (right)

Let $U$, $V$, and $W$ be (complex) $6 \times 6$ matrices and $\mu$, $\nu$, and $\eta$ be non-zero complex numbers. For the 6 directed arcs of $A(G_0)$ we define the following $6 \times 6$ matrices:

$$B(e_1) = U,\ \ B(e_2) = V,\ \ B(e_3) = W,$$

and $B(\overline{e}_1) = \mu B(e_1)^{\mathrm{T}}$, $B(\overline{e}_2) = \nu B(e_2)^{\mathrm{T}}$, and $B(\overline{e}_3) = \eta B(e_3)^{\mathrm{T}}$, i.e.,

$$B(\overline{e}_1) = \mu U^{\mathrm{T}},\ \ B(\overline{e}_2) = \nu V^{\mathrm{T}},\ \ B(\overline{e}_3) = \eta W^{\mathrm{T}}. \tag{5.1}$$

We demand that the following conditions are satisfied:

$$\sum_{i=1}^{3} \big(B(e_i)^* B(e_i) + B(\overline{e}_i)^* B(\overline{e}_i)\big) = I_6 \text{ and } \sum_{i=1}^{3} \big(B(e_i) B(e_i)^* + B(\overline{e}_i) B(\overline{e}_i)^*\big) = I_6. \tag{5.2}$$

For it, it is necessary and sufficient that $U$, $V$, $W$ and $\mu$, $\nu$, $\eta$ satisfy:

$$U^*U + V^*V + W^*W + |\mu|^2 (UU^*)^{\mathrm{T}} + |\nu|^2 (VV^*)^{\mathrm{T}} + |\eta|^2 (WW^*)^{\mathrm{T}} = I_6, \tag{5.3}$$

$$UU^* + VV^* + WW^* + |\mu|^2 (U^*U)^{\mathrm{T}} + |\nu|^2 (V^*V)^{\mathrm{T}} + |\eta|^2 (W^*W)^{\mathrm{T}} = I_6. \tag{5.4}$$

Notice that we can find many triples of matrices $\{U, V, W\}$ and parameters $\{\mu, \nu, \eta\}$ that satisfy the Eqs. (5.3) and (5.4). Here is an example: take $U$, $V$, and $W$ as the following form,

$$U = \begin{pmatrix} B & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ V = \begin{pmatrix} 0 & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & 0 \end{pmatrix},\ W = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & D \end{pmatrix}, \tag{5.5}$$

where $B$, $C$, and $D$ are $2 \times 2$ matrices, and the pairs $(B, \mu)$, $(C, \nu)$, and $(D, \eta)$ take the role of the pair $(B_1, \lambda)$ in (4.5)–(4.6).

The canonical basis of $l^2(\mathbb{T})$ is denoted by $\{|\mathbf{i}\rangle : \mathbf{i} \in \mathbb{T}\}$ and the coordinates of $\mathbb{C}^6$ will be indexed by $A(G_0)$, thereby it has a canonical basis $\{|e_i\rangle, |\bar{e}_i\rangle : i = 1, 2, 3\}$. Then, a basis for $\mathsf{h} = \oplus_{\mathbf{i}\in\mathbb{T}}\mathbb{C}^6$ is given by $\{e_{(\mathbf{i},e)} = |\mathbf{i}\rangle \otimes |e\rangle : \mathbf{i} \in \mathbb{T},\ e \in A(G_0)\}$. For each pair $(\mathbf{i}, e) \in \mathbb{T} \times A(G_0)$, define a linear operator $L_{(\mathbf{i},e)} : \mathsf{h} \to \mathsf{h}$ by

$$L_{(\mathbf{i},e)} := |\mathbf{i} + \widehat{\theta}(e)\rangle\langle \mathbf{i}| \otimes B(e).$$

We can check

$$\sum_{(\mathbf{i},e)} L^*_{(\mathbf{i},e)} L_{(\mathbf{i},e)} = I_{\mathsf{h}} \text{ and } \sum_{(\mathbf{i},e)} L_{(\mathbf{i},e)} L^*_{(\mathbf{i},e)} = I_{\mathsf{h}}. \tag{5.6}$$

As in the model on the cycle, the generator of QMS associated with the open quantum walk on the hexagonal torus is defined as follows: for $x \in \mathcal{A} := \oplus_{\mathbf{i}\in\mathbb{T}}\mathcal{B}(\mathbb{C}^6) \subset \mathcal{B}(\mathsf{h})$,

$$\begin{aligned}
\mathcal{L}(x) &:= -\frac{1}{2}\sum_{(\mathbf{i},e)}\Big(L^*_{(\mathbf{i},e)}L_{(\mathbf{i},e)}x - 2L^*_{(\mathbf{i},e)}xL_{(\mathbf{i},e)} + xL^*_{(\mathbf{i},e)}L_{(\mathbf{i},e)}\Big)\\
&= \sum_{(\mathbf{i},e)} L^*_{(\mathbf{i},e)}xL_{(\mathbf{i},e)} - x\\
&=: G^*x + \sum_{(\mathbf{i},e)} L^*_{(\mathbf{i},e)}xL_{(\mathbf{i},e)} + xG, \quad x \in \mathcal{B}(\mathsf{h}),
\end{aligned} \tag{5.7}$$

with $G = -\frac{1}{2}\sum_{(\mathbf{i},e)} L^*_{(\mathbf{i},e)}L_{(\mathbf{i},e)} = -\frac{1}{2}I_{\mathsf{h}}$. Notice that by (5.6), the state $\rho := \frac{1}{6mn}I_{\mathsf{h}}$ is an invariant state for the QMS.

### 5.3 Entropy production

In this subsection we compute the entropy production for the QMS with generator of (5.7). As in the previous section, $\theta$ is the conjugation operator with respect to the basis $\{e_{(\mathbf{i},e)} : (\mathbf{i}, e) \in \mathbb{T} \times A(G_0)\}$ and so the reversing operation $\Theta(a) := \theta a^* \theta$ is equal to taking a transpose $a^{\mathrm{T}}$ when the operators of $\mathcal{B}(\mathsf{h})$ are represented by the basis $\{e_{(\mathbf{i},e)}\}$. The definition of forward and backward two-point states are the same as given by Definition 3.1. Defining a unit vector

$$r = \frac{1}{\sqrt{6mn}} \sum_{(\mathbf{i},e)\in\mathbb{T}\times A(G_0)} e_{(\mathbf{i},e)} \otimes e_{(\mathbf{i},e)} \in \mathsf{h} \otimes \mathsf{h}$$

and a rank one projection $D := |r\rangle\langle r| \in \mathcal{B}(\mathsf{h}) \otimes \mathcal{B}(\mathsf{h})$, our construction of the QMS is to satisfy the assumption (**FBS**). Therefore, we can use the formula for the entropy production in Theorem 3.4.

For the computation of the entropy production we first need to check the orthogonality properties of the vectors $\{(\mathbb{1}\otimes L_{(\mathbf{i},e)})r\}_{(\mathbf{i},e)\in\mathbb{T}\times A(G_0)}$ and $\{(L_{(\mathbf{i},e)}\otimes \mathbb{1})r\}_{(\mathbf{i},e)\in\mathbb{T}\times A(G_0)}$ among themselves. Let us denote by $\chi$ the characteristic function $1_{\{e_3,\bar{e}_3\}}$ on $A(G_0)$. We can prove

**Lemma 5.1** *The following properties hold: for all* $\boldsymbol{i}, \boldsymbol{j} \in \mathbb{T}$ *and* $e, f \in A(G_0)$,

$$\langle (\mathbb{1} \otimes L_{(\boldsymbol{i},e)})r, (\mathbb{1} \otimes L_{(\boldsymbol{j},f)})r \rangle = \delta_{\boldsymbol{i},\boldsymbol{j}}\Big(\delta_{e,f}(1-\chi(e)(1-\chi(f)) + \chi(e)\chi(f)\Big)\tfrac{1}{6mn}\operatorname{tr}\big(B(e)^*B(f)\big),$$
$$\langle (L_{(\boldsymbol{i},e)} \otimes \mathbb{1})r, (L_{(\boldsymbol{j},f)} \otimes \mathbb{1})r \rangle = \delta_{\boldsymbol{i},\boldsymbol{j}}\Big(\delta_{e,f}(1-\chi(e)(1-\chi(f)) + \chi(e)\chi(f)\Big)\tfrac{1}{6mn}\operatorname{tr}\big(B(e)^*B(f)\big).$$

This lemma says particularly that the sets $\{(\mathbb{1} \otimes L_{(\mathbf{i},e)})r\}_{(\mathbf{i},e)\in\mathbb{T}\times A(G_0)}$ and $\{(L_{(\mathbf{i},e)} \otimes \mathbb{1})r\}_{(\mathbf{i},e)\in\mathbb{T}\times A(G_0)}$ are not orthogonal systems because $\langle (\mathbb{1}\otimes L_{(\mathbf{i},e_3)})r, (\mathbb{1}\otimes L_{(\mathbf{i},\bar{e}_3)})r\rangle \neq 0$ and $\langle (L_{(\mathbf{i},e_3)} \otimes \mathbb{1})r, (L_{(\mathbf{i},\bar{e}_3)} \otimes \mathbb{1})r\rangle \neq 0$ unless $\operatorname{tr}(W^*W^{\mathrm{T}}) = 0$. On the other hand, since we have to take logarithms of $\overrightarrow{\Phi}_*(D)$ and $\overleftarrow{\Phi}_*(D)$, we need to orthogonalize them. We therefore discuss the two cases separately.

### 5.3.1 The case $\operatorname{tr}(W^*W^{\mathrm{T}}) = 0$

For each $(\mathbf{i}, e) \in \mathbb{T} \times A(G_0)$, define the unit vectors:

$$u_{(\mathbf{i},e)} = \frac{1}{\sqrt{a_{(\mathbf{i},e)}}}(\mathbb{1} \otimes L_{(\mathbf{i},e)})r \text{ and } v_{(\mathbf{i},e)} = \frac{1}{\sqrt{b_{(\mathbf{i},e)}}}(L_{(\mathbf{i},e)} \otimes \mathbb{1})r, \tag{5.8}$$

where

$$a_{(\mathbf{i},e)} = \|(\mathbb{1}\otimes L_{(\mathbf{i},e)})r\|^2 = \frac{1}{6mn}\operatorname{tr}\big(B(e)^*B(e)\big),\ \ b_{(\mathbf{i},e)} = \|(L_{(\mathbf{i},e)} \otimes \mathbb{1})r\|^2$$
$$= \frac{1}{6mn}\operatorname{tr}\big(B(e)^*B(e)\big).$$

Then by Lemma 5.1, the sets $\{u_{(\mathbf{i},e)}\}_{(\mathbf{i},e)\in\mathbb{T}\times A(G_0)}$ and $\{v_{(\mathbf{i},e)}\}_{(\mathbf{i},e)\in\mathbb{T}\times A(G_0)}$ are orthonormal systems and we have the following representation

**Lemma 5.2** *Suppose that* $\operatorname{tr}(W^*W^{\mathrm{T}}) = 0$. *Then,*

$$\overrightarrow{\Phi}_*(D) = \sum_{(\boldsymbol{i},e)\in\mathbb{T}\times A(G_0)} a_{(\boldsymbol{i},e)}|u_{(\boldsymbol{i},e)}\rangle\langle u_{(\boldsymbol{i},e)}|,$$
$$\overleftarrow{\Phi}_*(D) = \sum_{(\boldsymbol{i},e)\in\mathbb{T}\times A(G_0)} b_{(\boldsymbol{i},e)}|v_{(\boldsymbol{i},e)}\rangle\langle v_{(\boldsymbol{i},e)}|.$$

For the computation of the entropy production, we need also

**Lemma 5.3** *For* $(\boldsymbol{i}, e), (\boldsymbol{j}, f) \in \mathbb{T} \times A(G_0)$,

$$\langle u_{(\boldsymbol{i},e)}, v_{(\boldsymbol{j},f)}\rangle = \frac{1}{6mn\sqrt{a_{(\boldsymbol{i},e)}b_{(\boldsymbol{j},f)}}}\delta_{\boldsymbol{i},\boldsymbol{j}+\widehat{\theta}(f)}\delta_{\boldsymbol{j},\boldsymbol{i}+\widehat{\theta}(e)}\operatorname{tr}\big(B(e)^*B(f)^{\mathrm{T}}\big).$$

***Proof*** Directly computing

$$\langle u_{(\mathbf{i},e)}, v_{(\mathbf{j},f)}\rangle = \frac{1}{6mn\sqrt{a_{(\mathbf{i},e)}b_{(\mathbf{j},f)}}}\delta_{\mathbf{i},\mathbf{j}+\widehat{\theta}(f)}\delta_{\mathbf{j},\mathbf{i}+\widehat{\theta}(e)}\sum_{g,g'\in A(G_0)}\langle g, B(f)g'\rangle\langle g, B(e)^*g'\rangle.$$

Here we have

$$\sum_{g,g'\in A(G_0)} \langle g, B(f)g'\rangle\langle g, B(e)^* g'\rangle = \sum_{g,g'\in A(G_0)} \langle g', B(f)^{\mathrm{T}} g\rangle\langle g, B(e)^* g'\rangle = \mathrm{tr}\big(B(e)^* B(f)^{\mathrm{T}}\big),$$

completing the proof. □

We are ready to compute the entropy production. Taking traces to both sides of (5.3) we get

$$(1+|\mu|^2)\mathrm{tr}\big(U^*U\big) + (1+|\nu|^2)\mathrm{tr}\big(V^*V\big) + (1+|\eta|^2)\mathrm{tr}\big(W^*W\big) = 6.$$

Let $\lambda_1, \lambda_2, \lambda_3 \in [0, 1]$ satisfying $\lambda_1 + \lambda_2 + \lambda_3 = 1$ be defined as

$$(1+|\mu|^2)\mathrm{tr}\big(U^*U\big) = 6\lambda_1,\ (1+|\nu|^2)\mathrm{tr}\big(V^*V\big) = 6\lambda_2,\ (1+|\eta|^2)\mathrm{tr}\big(W^*W\big) = 6\lambda_3. \tag{5.9}$$

We also introduce the parameters of Bernoulli distributions:

$$p_\mu = \tfrac{1}{1+|\mu|^2},\ q_\mu = \tfrac{|\mu|^2}{1+|\mu|^2},\ \ p_\nu = \tfrac{1}{1+|\nu|^2},\ q_\nu = \tfrac{|\nu|^2}{1+|\nu|^2},\ \ p_\eta = \tfrac{1}{1+|\eta|^2},\ q_\eta = \tfrac{|\eta|^2}{1+|\eta|^2}. \tag{5.10}$$

**Theorem 5.4** *Let $\mathcal{T} = (\mathcal{T}_t)_{t\ge 0}$ be the QMS with a generator given by (5.7). Suppose that* $\mathrm{tr}(W^*W^{\mathrm{T}}) = 0$. *Then the entropy production for the QMS is given by*

$$\begin{aligned}\boldsymbol{ep}(\mathcal{T},\rho) &= \lambda_1\Big(p_\mu \log\frac{p_\mu}{q_\mu} + q_\mu \log\frac{q_\mu}{p_\mu}\Big) + \lambda_2\Big(p_\nu \log\frac{p_\nu}{q_\nu} + q_\nu \log\frac{q_\nu}{p_\nu}\Big)\\ &\quad + \lambda_3\Big(p_\eta \log\frac{p_\eta}{q_\eta} + q_\eta \log\frac{q_\eta}{p_\eta}\Big).\end{aligned}$$

***Proof*** Using the entropy production formula in Theorem 3.4, by Lemmas 5.2 and 5.3, and the assumption $\mathrm{tr}(W^*W^{\mathrm{T}}) = 0$, we get

$$\begin{aligned}\mathbf{ep}(\mathcal{T},\rho) = \frac{1}{6}\Bigg(&\mathrm{tr}\big(U^*U\big)\log\frac{\mathrm{tr}\big(U^*U\big)}{6mn} + |\mu|^2\mathrm{tr}\big(U^*U\big)\log\frac{|\mu|^2\mathrm{tr}\big(U^*U\big)}{6mn}\\ &+\mathrm{tr}\big(V^*V\big)\log\frac{\mathrm{tr}\big(V^*V\big)}{6mn} + |\nu|^2\mathrm{tr}\big(V^*V\big)\log\frac{|\nu|^2\mathrm{tr}\big(V^*V\big)}{6mn}\\ &+\ \mathrm{tr}\big(W^*W\big)\log\frac{\mathrm{tr}\big(W^*W\big)}{6mn} + |\eta|^2\mathrm{tr}\big(W^*W\big)\log\frac{|\eta|^2\mathrm{tr}\big(W^*W\big)}{6mn}\Bigg)\\ &-\frac{1}{6}\Bigg(\mathrm{tr}\big(U^*U\big)\log\frac{|\mu|^2\mathrm{tr}\big(U^*U\big)}{6mn} + |\mu|^2\mathrm{tr}\big(U^*U\big)\log\frac{\mathrm{tr}\big(U^*U\big)}{6mn}\\ &+\mathrm{tr}\big(V^*V\big)\log\frac{|\nu|^2\mathrm{tr}\big(V^*V\big)}{6mn} + |\nu|^2\mathrm{tr}\big(V^*V\big)\log\frac{\mathrm{tr}\big(V^*V\big)}{6mn}\end{aligned}$$

$$
\begin{aligned}
&+\ \operatorname{tr}\big(W^*W\big)\log\frac{|\eta|^2\operatorname{tr}\big(W^*W\big)}{6mn}+|\eta|^2\operatorname{tr}\big(W^*W\big)\log\frac{\operatorname{tr}\big(W^*W\big)}{6mn}\Bigg)\\
&=\frac{1}{6}\Bigg(\operatorname{tr}\big(U^*U\big)\left(\log\frac{1}{|\mu|^2}+|\mu|^2\log|\mu|^2\right)\\
&\quad+\operatorname{tr}\big(V^*V\big)\left(\log\frac{1}{|\nu|^2}+|\nu|^2\log|\nu|^2\right)\\
&\quad+\ \operatorname{tr}\big(W^*W\big)\left(\log\frac{1}{|\eta|^2}+|\eta|^2\log|\eta|^2\right)\Bigg).
\end{aligned}
$$

Using (5.9) and (5.10) we can rewrite the last formula as given in the statement of the theorem. □

#### 5.3.2 The case $\operatorname{tr}(W^*W^{\mathrm{T}})\neq 0$

If $\operatorname{tr}(W^*W^{\mathrm{T}})\neq 0$, by Lemma 5.1, $\langle(\mathbb{1}\otimes L_{(\mathbf{i},e_3)})r,(\mathbb{1}\otimes L_{(\mathbf{i},\bar{e}_3)})r\rangle=\frac{\eta}{6mn}\operatorname{tr}(W^*W^{\mathrm{T}}))\neq 0$, i.e., $\{(\mathbb{1}\otimes L_{(\mathbf{i},e)})r\}_{(\mathbf{i},e)}$ is not an orthogonal system anymore. So, we need to orthogonalize to compute the logarithm of $\overrightarrow{\Phi}_*(D)$. Still we have to separately consider the cases according to whether the vectors $\{(\mathbb{1}\otimes L_{(\mathbf{i},e_3)})r,(\mathbb{1}\otimes L_{(\mathbf{i},\bar{e}_3)})r\}$ are linearly dependent or independent.

**Lemma 5.5** *The vectors* $\{(\mathbb{1}\otimes L_{(\mathbf{i},e_3)})r,(\mathbb{1}\otimes L_{(\mathbf{i},\bar{e}_3)})r\}$ *are linearly dependent if and only if* $|\operatorname{tr}(W^*W^{\mathrm{T}})|=\operatorname{tr}(W^*W)$.

***Proof*** Notice that two vectors $\{\mathbf{a},\mathbf{b}\}$ are linearly dependent if and only if $\|\mathbf{b}\|^2=|\langle\mathbf{b},\widehat{\mathbf{a}}\rangle|^2$, where $\widehat{\mathbf{a}}=\mathbf{a}/\|\mathbf{a}\|$. We use this and Lemma 5.1 to get the result. □

***Remark 5.6***

(i) By the Schwarz inequality we know $|\operatorname{tr}(W^*W^{\mathrm{T}})|\leq\operatorname{tr}(W^*W)$. Therefore, Lemma 5.5 equivalently says that the vectors $\{(\mathbb{1}\otimes L_{(\mathbf{i},e_3)})r,(\mathbb{1}\otimes L_{(\mathbf{i},\bar{e}_3)})r\}$ are linearly independent if and only if $|\operatorname{tr}(W^*W^{\mathrm{T}})|<\operatorname{tr}(W^*W)$.

(ii) The same is true for the pair of vectors $\{(L_{(\mathbf{i},e_3)}\otimes\mathbb{1})r,(L_{(\mathbf{i},\bar{e}_3)}\otimes\mathbb{1})r\}$.

**Case 1**: $0\neq|\operatorname{tr}(W^*W^{\mathrm{T}})|=\operatorname{tr}(W^*W)$.

As (5.8) we define the unit vectors $\{u_{(\mathbf{i},e)}=(\mathbb{1}\otimes L_{(\mathbf{i},e)})r/\sqrt{a_{(\mathbf{i},e)}}\}$ and $\{v_{(\mathbf{i},e)}=(L_{(\mathbf{i},e)}\otimes\mathbb{1})r/\sqrt{b_{(\mathbf{i},e)}}\}$. Since the vectors $\{(\mathbb{1}\otimes L_{(\mathbf{i},e_3)})r,(\mathbb{1}\otimes L_{(\mathbf{i},\bar{e}_3)})r\}$ are linearly dependent, noticing $a_{(\mathbf{i},\bar{e}_3)}=|\eta|^2a_{(\mathbf{i},e_3)}$ and $|u_{(\mathbf{i},\bar{e}_3)}\rangle\langle u_{(\mathbf{i},\bar{e}_3)}|=|u_{(\mathbf{i},e_3)}\rangle\langle u_{(\mathbf{i},e_3)}|$, we can write

$$
\begin{aligned}
&|(\mathbb{1}\otimes L_{(\mathbf{i},e_3)})r\rangle\langle(\mathbb{1}\otimes L_{(\mathbf{i},e_3)})r|+|(\mathbb{1}\otimes L_{(\mathbf{i},\bar{e}_3)})r\rangle\langle(\mathbb{1}\otimes L_{(\mathbf{i},\bar{e}_3)})r|\\
&\quad=a_{(\mathbf{i},e_3)}|u_{(\mathbf{i},e_3)}\rangle\langle u_{(\mathbf{i},e_3)}|+a_{(\mathbf{i},\bar{e}_3)}|u_{(\mathbf{i},\bar{e}_3)}\rangle\langle u_{(\mathbf{i},\bar{e}_3)}|\\
&\quad=(1+|\eta|^2)a_{(\mathbf{i},e_3)}|u_{(\mathbf{i},e_3)}\rangle\langle u_{(\mathbf{i},e_3)}|.
\end{aligned}
$$

Using this and similar computations for the pair $\{(L_{(\mathbf{i},e_3)}\otimes\mathbb{1})r,(L_{(\mathbf{i},\bar{e}_3)}\otimes\mathbb{1})r\}$, we have

**Lemma 5.7** *Suppose that* $0 \neq |tr(W^* W^{\mathrm{T}})| = tr(W^* W)$. *Then we have the representation*

$$\overrightarrow{\Phi}_*(D) = \sum_{\boldsymbol{i}\in\mathbb{T}} \left( \sum_{e\in A(G_0)\setminus\{e_3,\overline{e}_3\}} a_{(\boldsymbol{i},e)} |u_{(\boldsymbol{i},e)}\rangle\langle u_{(\boldsymbol{i},e)}| + (1+|\eta|^2) a_{(\boldsymbol{i},e_3)} |u_{(\boldsymbol{i},e_3)}\rangle\langle u_{(\boldsymbol{i},e_3)}| \right),$$

$$\overleftarrow{\Phi}_*(D) = \sum_{\boldsymbol{i}\in\mathbb{T}} \left( \sum_{e\in A(G_0)\setminus\{e_3,\overline{e}_3\}} b_{(\boldsymbol{i},e)} |v_{(\boldsymbol{i},e)}\rangle\langle v_{(\boldsymbol{i},e)}| + (1+|\eta|^2) b_{(\boldsymbol{i},e_3)} |v_{(\boldsymbol{i},e_3)}\rangle\langle v_{(\boldsymbol{i},e_3)}| \right).$$

We are ready to compute the entropy production.

**Theorem 5.8** *Suppose that* $0 \neq |tr(W^* W^{\mathrm{T}})| = tr(W^* W)$. *Then,*

$$\boldsymbol{ep}(\mathcal{T}, \rho) = \lambda_1 \left( p_\mu \log \frac{p_\mu}{q_\mu} + q_\mu \log \frac{q_\mu}{p_\mu} \right) + \lambda_2 \left( p_\nu \log \frac{p_\nu}{q_\nu} + q_\nu \log \frac{q_\nu}{p_\nu} \right),$$

*where the parameters are defined in* (5.9) *and* (5.10).

***Proof*** With the preparation in Lemma 5.7 we use the entropy production formula in Theorem 3.4 to get

$$\begin{aligned}
\mathbf{ep}(\mathcal{T}, \rho) = \frac{1}{6} \Bigg( & \mathrm{tr}\big(U^*U\big) \log \frac{\mathrm{tr}\big(U^*U\big)}{6mn} + |\mu|^2 \mathrm{tr}\big(U^*U\big) \log \frac{|\mu|^2 \mathrm{tr}\big(U^*U\big)}{6mn} \\
& + \mathrm{tr}\big(V^*V\big) \log \frac{\mathrm{tr}\big(V^*V\big)}{6mn} + |\nu|^2 \mathrm{tr}\big(V^*V\big) \log \frac{|\nu|^2 \mathrm{tr}\big(V^*V\big)}{6mn} \\
& + \ (1+|\eta|^2) \mathrm{tr}\big(W^*W\big) \log \frac{(1+|\eta|^2) \mathrm{tr}\big(W^*W\big)}{6mn} \Bigg) \\
& - \frac{1}{6} \Bigg( \mathrm{tr}\big(U^*U\big) \log \frac{|\mu|^2 \mathrm{tr}\big(U^*U\big)}{6mn} + |\mu|^2 \mathrm{tr}\big(U^*U\big) \log \frac{\mathrm{tr}\big(U^*U\big)}{6mn} \\
& + \mathrm{tr}\big(V^*V\big) \log \frac{|\nu|^2 \mathrm{tr}\big(V^*V\big)}{6mn} + |\nu|^2 \mathrm{tr}\big(V^*V\big) \log \frac{\mathrm{tr}\big(V^*V\big)}{6mn} \\
& + \ (1+|\eta|^2) \mathrm{tr}\big(W^*W\big) \log \frac{(1+|\eta|^2) \mathrm{tr}\big(W^*W\big)}{6mn} \Bigg) \\
= \frac{1}{6} \Bigg( & \mathrm{tr}\big(U^*U\big) \left( \log \frac{1}{|\mu|^2} + |\mu|^2 \log |\mu|^2 \right) \\
& + \mathrm{tr}\big(V^*V\big) \left( \log \frac{1}{|\nu|^2} + |\nu|^2 \log |\nu|^2 \right) \Bigg).
\end{aligned}$$

Using the parameters in (5.9) and (5.10) we get the result. ◻

**Case 2**: $0 \neq |\mathrm{tr}(W^* W^{\mathrm{T}})| < \mathrm{tr}(W^* W)$.

In this case by Lemma 5.5 the vectors $\{(\mathbb{1}\otimes L_{(\mathbf{i},e_3)})r,\ (\mathbb{1}\otimes L_{(\mathbf{i},\overline{e}_3)})r\}$ (and also $\{(L_{(\mathbf{i},e_3)}\otimes \mathbb{1})r,\ (L_{(\mathbf{i},\overline{e}_3)}\otimes \mathbb{1})r\}$) are linearly independent and they are not orthogonal to each other as we are assuming $\mathrm{tr}(W^*W^{\mathrm{T}})\neq 0$. Therefore we need to diagonalize the operator

$$|(\mathbb{1}\otimes L_{(\mathbf{i},e_3)})r\rangle\langle(\mathbb{1}\otimes L_{(\mathbf{i},e_3)})r| + |(\mathbb{1}\otimes L_{(\mathbf{i},\overline{e}_3)})r\rangle\langle(\mathbb{1}\otimes L_{(\mathbf{i},\overline{e}_3)})r|$$

for the functional calculus in the entropy production.

**Lemma 5.9** *Suppose that* $0\neq |tr(W^*W^{\mathrm{T}})| < tr(W^*W)$. *The operator*

$$|(\mathbb{1}\otimes L_{(\boldsymbol{i},e_3)})r\rangle\langle(\mathbb{1}\otimes L_{(\boldsymbol{i},e_3)})r| + |(\mathbb{1}\otimes L_{(\boldsymbol{i},\overline{e}_3)})r\rangle\langle(\mathbb{1}\otimes L_{(\boldsymbol{i},\overline{e}_3)})r|$$

*has eigenvalues*

$$\gamma_{\pm} = \frac{1}{12mn}\left((1+|\eta|^2)\mathrm{tr}\big(W^*W\big) \pm \sqrt{\big((1-|\eta|^2)\mathrm{tr}\big(W^*W\big)\big)^2 + \big(2|\eta|\mathrm{tr}\big(W^*W^{\mathrm{T}}\big)\big)^2}\right)$$

*with corresponding eigenvectors*

$$\begin{aligned}\xi_{(\boldsymbol{i},e_3)} &= a_+(\mathbb{1}\otimes L_{(\boldsymbol{i},e_3)})r + (\mathbb{1}\otimes L_{(\boldsymbol{i},\overline{e}_3)})r,\\ \xi_{(\boldsymbol{i},\overline{e}_3)} &= a_-(\mathbb{1}\otimes L_{(\boldsymbol{i},e_3)})r + (\mathbb{1}\otimes L_{(\boldsymbol{i},\overline{e}_3)})r,\end{aligned}$$

*where*

$$a_{\pm} = \frac{(1-|\eta|^2)\mathrm{tr}\big(W^*W\big) \pm \sqrt{\big((1-|\eta|^2)\mathrm{tr}\big(W^*W\big)\big)^2 + \big(2|\eta|\mathrm{tr}\big(W^*W^{\mathrm{T}}\big)\big)^2}}{2\overline{\eta}\,\mathrm{tr}\big(W^*W^{\mathrm{T}}\big)}. \tag{5.11}$$

*Similarly, the operator*

$$|(L_{(\boldsymbol{i},e_3)}\otimes \mathbb{1})r\rangle\langle(L_{(\boldsymbol{i},e_3)}\otimes \mathbb{1})r| + |(L_{(\boldsymbol{i},\overline{e}_3)}\otimes \mathbb{1})r\rangle\langle(L_{(\boldsymbol{i},\overline{e}_3)}\otimes \mathbb{1}\otimes)r|$$

*has the same eigenvalues* $\gamma_{\pm}$ *with corresponding eigenvectors*

$$\begin{aligned}\zeta_{(\boldsymbol{i},e_3)} &= a_+(L_{(\boldsymbol{i},e_3)}\otimes \mathbb{1})r + (L_{(\boldsymbol{i},\overline{e}_3)}\otimes \mathbb{1})r,\\ \zeta_{(\boldsymbol{i},\overline{e}_3)} &= a_-(L_{(\boldsymbol{i},e_3)}\otimes \mathbb{1})r + (L_{(\boldsymbol{i},\overline{e}_3)}\otimes \mathbb{1})r.\end{aligned}$$

***Proof*** It is just a two-dimensional eigenvalue problem. □

Now for $\mathbf{i}\in\mathbb{T}$ and $e\in A(G_0)\backslash\{e_3,\overline{e}_3\}$, as before, we let

$$u_{(\mathbf{i},e)} = \frac{1}{\sqrt{a_{(\mathbf{i},e)}}}(\mathbb{1}\otimes L_{(\mathbf{i},e)})r \text{ and } v_{(\mathbf{i},e)} = \frac{1}{\sqrt{b_{(\mathbf{i},e)}}}(L_{(\mathbf{i},e)}\otimes \mathbb{1})r$$

with $a_{(\mathbf{i},e)} = \|(\mathbb{1}\otimes L_{(\mathbf{i},e)})r\|^2$, $b_{(\mathbf{i},e)} = \|(L_{(\mathbf{i},e)}\otimes \mathbb{1})r\|^2$. Denoting $\widehat{\xi}_{(\mathbf{i},e_3)} = \xi_{(\mathbf{i},e_3)}/\|\xi_{(\mathbf{i},e_3)}\|$ and $\widehat{\zeta}_{(\mathbf{i},e_3)} = \zeta_{(\mathbf{i},e_3)}/\|\zeta_{(\mathbf{i},e_3)}\|$, the systems $\{u_{(\mathbf{i},e)}\}_{(\mathbf{i},e)\in\mathbb{T}\times A(G_0)\backslash\{e_3,\overline{e}_3\}} \cup \{\widehat{\xi}_{(\mathbf{i},e)}\}_{(\mathbf{i},e)\in\mathbb{T}\times\{e_3,\overline{e}_3\}}$ and $\{v_{(\mathbf{i},e)}\}_{(\mathbf{i},e)\in\mathbb{T}\times A(G_0)\backslash\{e_3,\overline{e}_3\}} \cup \{\widehat{\zeta}_{(\mathbf{i},e)}\}_{(\mathbf{i},e)\in\mathbb{T}\times\{e_3,\overline{e}_3\}}$ are, respectively, orthonormal systems in $\mathsf{h}\otimes\mathsf{h}$, and we have the following representation.

**Lemma 5.10** *It holds that*

$$
\begin{aligned}
&\overrightarrow{\Phi}_*(D)\\
&=\sum_{i\in\mathbb{T}}\Big(\sum_{e\in A(G_0)\setminus\{e_3,\overline{e}_3\}} a_{(i,e)}|u_{(i,e)}\rangle\langle u_{(i,e)}|+\gamma_+|\widehat{\xi}_{(i,e_3)}\rangle\langle\widehat{\xi}_{(i,e_3)}|+\gamma_-|\widehat{\xi}_{(i,\overline{e}_3)}\rangle\langle\widehat{\xi}_{(i,\overline{e}_3)}|\Big),\\
&\overleftarrow{\Phi}_*(D)\\
&=\sum_{i\in\mathbb{T}}\Big(\sum_{e\in A(G_0)\setminus\{e_3,\overline{e}_3\}} b_{(i,e)}|v_{(i,e)}\rangle\langle v_{(i,e)}|+\gamma_+|\widehat{\zeta}_{(i,e_3)}\rangle\langle\widehat{\zeta}_{(i,e_3)}|+\gamma_-|\widehat{\zeta}_{(i,\overline{e}_3)}\rangle\langle\widehat{\zeta}_{(i,\overline{e}_3)}|\Big).
\end{aligned}
$$

**Lemma 5.11** *We have*

$$
\begin{aligned}
&\langle\widehat{\xi}_{(i,e_3)},\widehat{\zeta}_{(i,e_3)}\rangle\\
&=\frac{1}{6mn\|\xi_{(i,e_3)}\|\,\|\zeta_{(i,e_3)}\|}\Big((|a_+|^2+|\eta|^2)\mathrm{tr}\big(W^*W^{\mathrm{T}}\big)+(\overline{a}_+\eta+a_+\overline{\eta})\mathrm{tr}\big(W^*W\big)\Big),\\
&\langle\widehat{\xi}_{(i,e_3)},\widehat{\zeta}_{(i,\overline{e}_3)}\rangle\\
&=\frac{1}{6mn\|\xi_{(i,e_3)}\|\,\|\zeta_{(i,\overline{e}_3)}\|}\Big((\overline{a}_+a_-+|\eta|^2)\mathrm{tr}\big(W^*W^{\mathrm{T}}\big)+(\overline{a}_+\eta+a_-\overline{\eta})\mathrm{tr}\big(W^*W\big)\Big),\\
&\langle\widehat{\xi}_{(i,\overline{e}_3)},\widehat{\zeta}_{(i,e_3)}\rangle\\
&=\frac{1}{6mn\|\xi_{(i,\overline{e}_3)}\|\,\|\zeta_{(i,e_3)}\|}\Big((\overline{a}_-a_++|\eta|^2)\mathrm{tr}\big(W^*W^{\mathrm{T}}\big)+(\overline{a}_-\eta+a_+\overline{\eta})\mathrm{tr}\big(W^*W\big)\Big),\\
&\langle\widehat{\xi}_{(i,\overline{e}_3)},\widehat{\zeta}_{(i,\overline{e}_3)}\rangle\\
&=\frac{1}{6mn\|\xi_{(i,\overline{e}_3)}\|\,\|\zeta_{(i,\overline{e}_3)}\|}\Big((|a_-|^2+|\eta|^2)\mathrm{tr}\big(W^*W^{\mathrm{T}}\big)+(\overline{a}_-\eta+a_-\overline{\eta})\mathrm{tr}\big(W^*W\big)\Big),
\end{aligned}
$$

*with the vector norms*

$$
\begin{aligned}
\|\xi_{(i,e_3)}\|^2&=\frac{1}{6mn}\Big((|a_+|^2+|\eta|^2)\mathrm{tr}\big(W^*W\big)+(\overline{a}_+\eta+a_+\overline{\eta})\mathrm{tr}\big(W^*W^{\mathrm{T}}\big)\Big),\\
\|\xi_{(i,\overline{e}_3)}\|^2&=\frac{1}{6mn}\Big((|a_-|^2+|\eta|^2)\mathrm{tr}\big(W^*W\big)+(\overline{a}_-\eta+a_-\overline{\eta})\mathrm{tr}\big(W^*W^{\mathrm{T}}\big)\Big),\\
\|\zeta_{(i,e_3)}\|^2&=\|\xi_{(i,e_3)}\|^2,\ \ \|\zeta_{(i,\overline{e}_3)}\|^2=\|\xi_{(i,\overline{e}_3)}\|^2.
\end{aligned}
$$

As a final preparation, we will see that the vector $\xi_{(\mathbf{i},e_3)}$ and $\xi_{(\mathbf{i},\overline{e}_3)}$ lie in the space spanned by the vectors $\{\zeta_{(\mathbf{i},e_3)},\zeta_{(\mathbf{i},\overline{e}_3)}\}$.

**Lemma 5.12** *We have*

$$
\mathrm{span}\{\xi_{(i,e_3)},\xi_{(i,\overline{e}_3)}\}=\mathrm{span}\{\zeta_{(i,e_3)},\zeta_{(i,\overline{e}_3)}\}.
$$

***Proof*** It is equivalent to show

$$
\mathrm{span}\{(\mathbb{1}\otimes L_{(\mathbf{i},e_3)})r,\ (\mathbb{1}\otimes L_{(\mathbf{i},\overline{e}_3)})r\}=\mathrm{span}\{(L_{(\mathbf{i},e_3)}\otimes\mathbb{1})r,\ (L_{(\mathbf{i},\overline{e}_3)}\otimes\mathbb{1})r\}.
$$

In fact, we will see that $(\mathbb{1}\otimes L_{(\mathbf{i},e_3)})r = \frac{1}{\eta}(L_{(\mathbf{i},\overline{e}_3)}\otimes \mathbb{1})r$ and $(\mathbb{1}\otimes L_{(\mathbf{i},\overline{e}_3)})r = \eta(L_{(\mathbf{i},e_3)}\otimes \mathbb{1})r$. Let us denote the matrix components of $W$ with respect to the canonical basis $\{|e_i\rangle, |\overline{e}_i\rangle : i = 1,2,3\}$ of $\mathbb{C}^6$ by $W_{e,f} := \langle e, Wf\rangle$ for $|e\rangle, |f\rangle \in \{|e_i\rangle, |\overline{e}_i\rangle : i = 1,2,3\}$. Notice that $W_{e,f} = W^{\mathrm{T}}_{f,e}$. By definition,

$$\begin{aligned}
\sqrt{6mn}(\mathbb{1}\otimes L_{(\mathbf{i},e_3)})r &= \Big(\mathbb{1}\otimes\big(|\mathbf{i}+\widehat{\theta}(e_3)\rangle\langle\mathbf{i}|\otimes B(e_3)\big)\Big)\sum_{(\mathbf{j},e)}(|\mathbf{j}\rangle\otimes|e\rangle)\otimes(|\mathbf{j}\rangle\otimes|e\rangle)\\
&= (\mathbb{1}\otimes(|\mathbf{i}\rangle\langle\mathbf{i}|\otimes W))\sum_{(\mathbf{j},e)}(|\mathbf{j}\rangle\otimes|e\rangle)\otimes(|\mathbf{j}\rangle\otimes|e\rangle)\\
&= \sum_{e}(|\mathbf{i}\rangle\otimes|e\rangle)\otimes(|\mathbf{i}\rangle\otimes|We\rangle)\\
&= \sum_{e,f}W_{f,e}\,(|\mathbf{i}\rangle\otimes|e\rangle)\otimes(|\mathbf{i}\rangle\otimes|f\rangle)\\
&= \sum_{f}\Big(|\mathbf{i}\rangle\otimes|W^{\mathrm{T}}f\rangle\Big)\otimes(|\mathbf{i}\rangle\otimes|f\rangle)\\
&= \frac{1}{\eta}\sqrt{6mn}(L_{(\mathbf{i},\overline{e}_3)}\otimes\mathbb{1})r.
\end{aligned}$$

This shows that $(\mathbb{1}\otimes L_{(\mathbf{i},e_3)})r \in \mathrm{span}\{(L_{(\mathbf{i},e_3)}\otimes\mathbb{1})r, (L_{(\mathbf{i},\overline{e}_3)}\otimes\mathbb{1})r\}$. By similar computations we show the statement of the lemma. □

We can now compute the entropy production. We define the parameters as in (5.9) and (5.10). One more pair of Bernoulli parameters is defined as $p_\gamma := \gamma_+/(\gamma_+ + \gamma_-)$ and $q_\gamma := \gamma_-/(\gamma_+ + \gamma_-)$.

**Theorem 5.13** *Suppose that* $0 \neq |tr(W^*W^{\mathrm{T}})| < tr(W^*W)$. *Then,*

$$\begin{aligned}
\boldsymbol{ep}(\mathcal{T},\rho) = \lambda_1\left(p_\mu\log\frac{p_\mu}{q_\mu} + q_\mu\log\frac{q_\mu}{p_\mu}\right) + \lambda_2\Big(p_\nu\log\frac{p_\nu}{q_\nu} + q_\nu\log\frac{q_\nu}{p_\nu}\Big)\\
+\lambda_3\left(p_\gamma\log\frac{p_\gamma}{q_\gamma} + q_\gamma\log\frac{q_\gamma}{p_\gamma}\right)|\langle\widehat{\xi}_{(\boldsymbol{i},e_3)},\widehat{\zeta}_{(\boldsymbol{i},\overline{e}_3)}\rangle|^2.
\end{aligned}$$

***Proof*** Using Theorem 3.4 we compute as in the proof of Theorem 5.8 to get

$$\begin{aligned}
\mathbf{ep}(\mathcal{T},\rho) = \frac{1}{6}\Big(&\mathrm{tr}\big(U^*U\big)\Big(\log\frac{1}{|\mu|^2} + |\mu|^2\log|\mu|^2\Big)\\
&+\mathrm{tr}\big(V^*V\big)\Big(\log\frac{1}{|\nu|^2} + |\nu|^2\log|\nu|^2\Big)\Big)\\
&+mn\Big(\gamma_+\log\gamma_+ + \gamma_-\log\gamma_-\Big)\\
&-mn\Big(\gamma_+\log\gamma_+|\langle\widehat{\xi}_{(\mathbf{i},e_3)},\widehat{\zeta}_{(\mathbf{i},e_3)}\rangle|^2 + \gamma_+\log\gamma_-|\langle\widehat{\xi}_{(\mathbf{i},e_3)},\widehat{\zeta}_{(\mathbf{i},\overline{e}_3)}\rangle|^2\\
&+\gamma_-\log\gamma_+|\langle\widehat{\xi}_{(\mathbf{i},\overline{e}_3)},\widehat{\zeta}_{(\mathbf{i},e_3)}\rangle|^2 + \gamma_-\log\gamma_-|\langle\widehat{\xi}_{(\mathbf{i},\overline{e}_3)},\widehat{\zeta}_{(\mathbf{i},\overline{e}_3)}\rangle|^2\Big).
\end{aligned}$$

By Lemma 5.12 we have the identity

$$1 = \|\widehat{\xi}_{(\mathbf{i},e_3)}\|^2 = |\langle \widehat{\xi}_{(\mathbf{i},e_3)}, \widehat{\zeta}_{(\mathbf{i},e_3)}\rangle|^2 + |\langle \widehat{\xi}_{(\mathbf{i},e_3)}, \widehat{\zeta}_{(\mathbf{i},\bar{e}_3)}\rangle|^2.$$

Hence

$$|\langle \widehat{\xi}_{(\mathbf{i},e_3)}, \widehat{\zeta}_{(\mathbf{i},e_3)}\rangle|^2 = 1 - |\langle \widehat{\xi}_{(\mathbf{i},e_3)}, \widehat{\zeta}_{(\mathbf{i},\bar{e}_3)}\rangle|^2.$$

Similarly, we have the identity

$$|\langle \widehat{\xi}_{(\mathbf{i},\bar{e}_3)}, \widehat{\zeta}_{(\mathbf{i},\bar{e}_3)}\rangle|^2 = 1 - |\langle \widehat{\xi}_{(\mathbf{i},\bar{e}_3)}, \widehat{\zeta}_{(\mathbf{i},e_3)}\rangle|^2.$$

We implement these identities to the formula above and use the identity $|\langle \widehat{\xi}_{(\mathbf{i},\bar{e}_3)}, \widehat{\zeta}_{(\mathbf{i},e_3)}\rangle|^2 = |\langle \widehat{\xi}_{(\mathbf{i},e_3)}, \widehat{\zeta}_{(\mathbf{i},\bar{e}_3)}\rangle|^2$ coming from Lemma 5.11, to get

$$\begin{aligned}\mathbf{ep}(\mathcal{T},\rho) = \frac{1}{6}\Big(&\mathrm{tr}\big(U^*U\big)\Big(\log\frac{1}{|\mu|^2} + |\mu|^2\log|\mu|^2\Big)\\ &+\mathrm{tr}\big(V^*V\big)\Big(\log\frac{1}{|\nu|^2} + |\nu|^2\log|\nu|^2\Big)\Big)\\ &+mn\Big(\gamma_+\log\frac{\gamma_+}{\gamma_-} + \gamma_-\log\frac{\gamma_-}{\gamma_+}\Big)|\langle \widehat{\xi}_{(\mathbf{i},e_3)}, \widehat{\zeta}_{(\mathbf{i},\bar{e}_3)}\rangle|^2.\end{aligned}$$

Using the already defined parameters the last formula changes into the one given in the statement of the theorem. The proof is completed. □

***Remark 5.14*** The different formulas for the entropy production in Theorems 5.4, 5.8, and 5.13 resulted only from the internal movement, or the rotation. It depends whether the projections that taking charge of the internal movement are linearly dependent or independent and also whether they are orthogonal to each other or not. In the following subsection, we will see that when the internal movement is defined by the rotation matrix, the rotation angles determine the cases (see Proposition 5.16).

## 5.4 Examples

In this subsection we provide with some examples for which the different cases in the previous subsection occur. We focus on the form of the matrix $W$, which was used to define $B(e_3) = W$ and $B(\bar{e}_3) = \eta W^{\mathrm{T}}$. For simplicity, let us consider the model with the matrices of the form of (5.5) together with an idea in Example 4.5. To say more concretely, let $R = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$, $-\pi < \theta \le \pi$, be the rotation matrix by an angle $\theta$. Then, let $D = \frac{1}{\sqrt{1+|\eta|^2}}R$ and $W$ is defined as in (5.5).

**Lemma 5.15** *We have the following relations:*

(i) $tr(W^*W^{\mathrm{T}}) = 0$ *if and only if* $\cos^2\theta = \sin^2\theta$, *i.e.,* $\theta \in \{\pm\pi/4, \pm 3\pi/4\}$*;*
(ii) $|tr(W^*W^{\mathrm{T}})| = tr(W^*W)$ *if and only if* $\cos\theta\sin\theta = 0$, *i.e.,* $\theta \in \{0, \pi, \pm\pi/2\}$, *otherwise* $|tr(W^*W^{\mathrm{T}})| < tr(W^*W)$.

***Proof*** The proof follows easily from the direct computation. □

**Proposition 5.16** *Consider a QMS associated with an OQW on the hexagonal torus whose generator is defined by the matrices $U$, $V$, and $W$ satisfying (5.3) and (5.4). Suppose further that the matrix $W$ is defined by using a $2 \times 2$ rotation matrix $R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ as introduced in the beginning of this subsection. According to the rotation angle $\theta$ we have:*

(i) *if $\theta \in \{\pm\pi/4, \pm 3\pi/4\}$, then the entropy production is given by the formula in Theorem 5.4,*
(ii) *if $\theta \in \{0, \pi, \pm\pi/2\}$, then the entropy production is given by the formula in Theorem 5.8,*
(iii) *if $\theta \notin \{\pm\pi/4, \pm 3\pi/4, 0, \pi, \pm\pi/2\}$, then the entropy production is given by the formula in Theorem 5.13.*

***Proof*** The proof follows from Lemma 5.15. □

## 6 No-existence of entropy production and SQDB-Θ

In this Section we investigate the relationship between no-existence of entropy production and SQDB-Θ. We separately discuss the QMSs for the OQWs on the regular and Hexagonal lattices with periodic boundary conditions.

### 6.1 QMS associated with OQW on a torus

Let us consider the model on the regular integer lattice, typically the model on the torus discussed in Sect. 4. We start with a lemma.

**Lemma 6.1** *Let $\mathcal{T}$ be a QMS associated with OQW on a torus with a generator in (4.18). Then, $\mathcal{T}$ satisfies SQDB-Θ if and only if the parameters $\mu$ and $\nu$ in (4.12) satisfy $|\mu| = |\nu| = 1$.*

***Proof*** Suppose that $|\mu| = |\nu| = 1$. We will show that the conditions (i) and (ii) in Theorem 2.4 hold, thereby the SQDB-Θ is satisfied. Since the operators $G$ and $\rho$ are constant multiples of the identity, (i) holds trivially. We check the condition (ii). Recall that $\Theta(L_k) = L_k^{\mathrm{T}}$. From (4.12) and the condition $|\mu| = |\nu| = 1$, we see that

$$B_1^{\mathrm{T}} = \overline{\mu} B_{-1},\ B_{-1}^{\mathrm{T}} = \mu B_1,\ B_2^{\mathrm{T}} = \overline{\nu} B_{-2},\ B_{-2}^{\mathrm{T}} = \nu B_2.$$

Therefore, from (4.16) we see that

$$\begin{aligned} L_{(\mathbf{i},\alpha)}^{\mathrm{T}} &= |\mathbf{i}\rangle\langle \mathbf{i} + \widehat{\theta}(e_\alpha)| \otimes B_\alpha^{\mathrm{T}} \\ &= w_\alpha |(\mathbf{i} + \widehat{\theta}(e_\alpha)) + \widehat{\theta}(e_{-\alpha})\rangle\langle \mathbf{i} + \widehat{\theta}(e_\alpha)| \otimes B_{-\alpha} \\ &= w_\alpha L_{(\mathbf{i}+\widehat{\theta}(e_\alpha)), -\alpha)}. \end{aligned}$$

where $(w_1, w_{-1}, w_2, w_{-2}) = (\overline{\mu}, \mu, \overline{\nu}, \nu)$. Obviously one can write

$$L^{\mathrm{T}}_{(\mathbf{i},\alpha)} = \sum_{(\mathbf{j},\beta)} u_{(\mathbf{i},\alpha)(\mathbf{j},\beta)} L_{(\mathbf{j},\mathbf{fi})},$$

with $u_{(\mathbf{i},\alpha)(\mathbf{j},\beta)} = w_\alpha \delta_{\mathbf{i}+\widehat{\theta}(e_\alpha),\mathbf{j}} \delta_{-\alpha,\beta}$, which constitute a self-adjoint unitary matrix on the Hilbert space $\mathsf{k}$ whose orthonormal basis is indexed by $\{(\mathbf{i},\alpha) : \mathbf{i} \in \mathbb{T}_{(m,n)}, \alpha \in \{\pm 1, \pm 2\}\}$.

Now suppose that $\mathcal{T}$ satisfies SQDB-$\Theta$. There is a set $(L'_{(\mathbf{i},\alpha)})_{(\mathbf{i},\alpha)}$ of bounded operators in $\mathcal{B}(\mathsf{h})$ that satisfy the relation (ii) of Theorem 2.4. That is, there is a self-adjoint unitary matrix $U = (u_{(\mathbf{i},\alpha)(\mathbf{j},\beta)})$ on $\mathsf{k}$ such that

$$L'^{\mathrm{T}}_{(\mathbf{i},\alpha)} = \sum_{(\mathbf{j},\beta)} u_{(\mathbf{i},\alpha)(\mathbf{j},\beta)} L'_{(\mathbf{j},\mathbf{fi})}. \tag{6.1}$$

On the other hand, by condition (2.1), there is a unitary $V = (v_{(\mathbf{i},\alpha)(\mathbf{j},\beta)})$ on $\mathsf{k}$ such that

$$L_{(\mathbf{i},\alpha)} = \sum_{(\mathbf{j},\beta)} v_{(\mathbf{i},\alpha)(\mathbf{j},\beta)} L'_{(\mathbf{j},\mathbf{fi})}. \tag{6.2}$$

Combining (6.1) and (6.2), we see that

$$L^{\mathrm{T}}_{(\mathbf{i},\alpha)} = \sum_{(\mathbf{j},\beta)} (VUV^*)_{(\mathbf{i},\alpha)(\mathbf{j},\beta)} L_{(\mathbf{j},\mathbf{fi})},$$

and $VUV^*$ is a self-adjoint unitary matrix on $\mathsf{k}$. We already know, however, the relation

$$L^{\mathrm{T}}_{(\mathbf{i},\alpha)} = \sum_{(\mathbf{j},\beta)} w'_{(\mathbf{i},\alpha)(\mathbf{j},\beta)} L_{(\mathbf{j},\mathbf{fi})},$$

where $w'_{(\mathbf{i},\alpha)(\mathbf{j},\beta)} = w'_\alpha \delta_{\mathbf{i}+\widehat{\theta}(e_\alpha),\mathbf{j}} \delta_{-\alpha,\beta}$ with $(w'_1, w'_{-1}, w'_2, w'_{-2}) = (\mu^{-1}, \mu, \nu^{-1}, \nu)$. In order that the matrix $W' = (w'_{(\mathbf{i},\alpha)(\mathbf{j},\beta)})$ to be a self-adjoint unitary (as it is for $VUV^*$) it must hold that $\mu^{-1} = \overline{\mu}$ and $\nu^{-1} = \nu$, equivalently $|\mu| = |\nu| = 1$. The proof is completed. □

**Theorem 6.2** *Let $\mathcal{T}$ be a QMS associated with OQW on a torus with a generator in (4.18). Then, the entropy production is zero if and only if $\mathcal{T}$ satisfies SQDB-$\Theta$.*

***Proof*** From Theorem 4.7, we see that the entropy production of the QMS is zero if and only if $p_\mu = q_\mu = 1/2$ and $p_\nu = q_\nu = 1/2$. On the other hand the latter conditions are equivalent to $|\mu| = |\nu| = 1$. The result follows from Lemma 6.1. □

The above result also holds for the model on the cycle as well as it can be extended to multi-dimensional integer lattices.

### 6.2 QMS associated with OQW on a hexagonal torus

Recall that the generator of the model is given in (5.7). As in the model on a torus, we have the following characterization.

**Lemma 6.3** *Let $\mathcal{T}$ be a QMS associated with OQW on the hexagonal torus with a generator in* (5.7)*. Then, $\mathcal{T}$ satisfies SQDB-$\Theta$ if and only if the parameters $\mu$, $\nu$, and $\eta$ in* (5.1) *satisfy $|\mu| = |\nu| = |\eta| = 1$.*

***Proof*** One can show this lemma by the similar methods used in the proof of Lemma 6.1. □

We would like to see the relationship between no-existence of entropy production and SQDB-$\Theta$. Notice that for this model, as we have seen in Sect. 5, the values of entropy production differ case by case. So, here we also consider each case separately.

**Case 1**: $\mathrm{tr}(W^*W^{\mathrm{T}}) = 0$.
In this case the entropy production formula is given by Theorem 5.4. The formula says that the entropy production is equal to zero if and only if the parameters $\mu, \nu, \eta$ in (5.1) satisfy $|\mu| = |\nu| = |\eta| = 1$. Together with Lemma 6.3, we conclude that the entropy production is zero if and only if the SQDB-$\Theta$ holds.

**Case 2**: $0 \neq |\mathrm{tr}(W^*W^{\mathrm{T}})| = \mathrm{tr}(W^*W)$.
In this case, the entropy production is given by Theorem 5.8. By it, the entropy production is zero if and only if $|\mu| = |\nu| = 1$, but the value of $\eta$ may be arbitrary. Therefore, by Lemma 6.3, we conclude that if the SQDB-$\Theta$ holds, then the entropy production is zero. However, the converse is not true: we may have zero entropy production while the SQDB-$\Theta$ fails.

**Case 3**: $0 \neq |\mathrm{tr}(W^*W^{\mathrm{T}})| < \mathrm{tr}(W^*W)$.
In this case the entropy production is given by Theorem 5.13. Since $p_\gamma \neq q_\gamma$ in the formula we see that the entropy production is zero if and only if

$$|\mu| = |\nu| = 1 \text{ and } \langle \widehat{\xi}_{(\mathbf{i},e_3)}, \widehat{\zeta}_{(\mathbf{i},\overline{e}_3)} \rangle = 0.$$

But as the Lemma 6.4 below shows the latter condition is equivalent to $|\eta| = 1$. Therefore, as in the Case 1, we conclude that the entropy production is zero if and only if the SQDB-$\Theta$ holds.

**Lemma 6.4** *In the case $0 \neq |tr(W^*W^{\mathrm{T}})| < tr(W^*W)$, we have $\langle \widehat{\xi}_{(\boldsymbol{i},e_3)}, \widehat{\zeta}_{(\boldsymbol{i},\overline{e}_3)} \rangle = 0$ if and only if $|\eta| = 1$.*

***Proof*** By Lemma 5.11, $\langle \widehat{\xi}_{(\mathbf{i},e_3)}, \widehat{\zeta}_{(\mathbf{i},\overline{e}_3)} \rangle = 0$ if and only if

$$(\overline{a}_+ a_- + |\eta|^2)\mathrm{tr}\big(W^*W^{\mathrm{T}}\big) + (\overline{a}_+\eta + a_-\overline{\eta})\mathrm{tr}\big(W^*W\big) = 0. \tag{6.3}$$

From (5.11), we directly compute to see

$$\overline{a}_+ a_- = -1 \text{ and } \overline{a}_+\eta + a_-\overline{\eta} = \frac{(1-|\eta|^2)\mathrm{tr}\big(W^*W\big)}{\mathrm{tr}\big(W^*W^{\mathrm{T}}\big)}.$$

Implementing into (6.3), we conclude

$$\langle \widehat{\xi}_{(\mathbf{i},e_3)}, \widehat{\zeta}_{(\mathbf{i},\bar{e}_3)} \rangle = 0 \text{ if and only if } (1-|\eta|^2)(\mathrm{tr}\big(W^*W\big)^2 - \mathrm{tr}\big(W^*W^{\mathrm{T}}\big)^2) = 0,$$

and the result follows since $|\mathrm{tr}(W^*W^{\mathrm{T}})| < \mathrm{tr}(W^*W)$. □

Summarizing, we see that the QMS associated with the OQW on the hexagonal torus has no entropy production if the SQDB-Θ holds. The converse is true in certain cases but there is also a case where it is not true. It is worth to notice that such a phenomenon (zero entropy production without SQDB-Θ) also occurs in some other model (see [9, Example 7.3]).

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